8 The energy momentum tensor

We have now developed the necessary mathematical machinery to describe the curvature of spacetime in a manner which can be expressed *tensorially*, i.e. independently of one's choice of coordinate system. In Section 9 we will define the *Riemann Christoffel tensor*, which provides this geometrical description of curvature. Before we do that, however, in this section we first consider the *source* of this spacetime curvature – the gravitating matter content of spacetime.

The energy momentum tensor (also known as the stress energy tensor) describes the presence and motion of gravitating matter. In this section we will discuss the energy momentum tensor for the particular case of a 'perfect fluid', which is a mathematical idealisation but one which is a good approximate description of the gravitating matter in many astrophysical situations. We will derive the form of the energy momentum tensor for a perfect fluid in special relativity; this form is easily generalised to GR. Finally we will prove an important result about the *divergence* of the energy momentum tensor.

8.1 Perfect fluids

Many Newtonian gravitational problems can be considered simply as the interaction of a small number of point-like massive particles – e.g. the 2-body interaction between the Earth and the Moon can be accurately modelled in this way. Even in Newtonian theory, however, there are many contexts (e.g. the motion of stars in the Galaxy) where the number of gravitating 'particles' is too large to follow their individual trajectories. Instead we treat the system as a smooth continuum, or **fluid**, and describe its behaviour in terms of the locally averaged properties (e.g. the *density*, *velocity* or *temperature*) of the particles in each **fluid element** – by which we mean a small region of the fluid surrounding some point in the continuum within which the behaviour of the particles is fairly homogeneous.

This fluid description is also useful for many-particle systems in special relativity, although we must be careful about defining quantities such as density and velocity which are frame-dependent – i.e. we need to find a *covariant* description of the fluid (which, we will see, is why we require a *tensor* to describe the gravitating matter).

The simplest type of relativistic fluid is known as 'dust'. To a physicist, a fluid element of dust means a collection of particles which are all at rest with respect to some Lorentz frame. Many textbooks (including Schutz) refer to this Lorentz frame as the *momentarily comoving rest frame* (MCRF) of the fluid element. This name helps to reinforce the point that the fluid element as a whole may possess a bulk motion with respect to the rest of the fluid, and indeed this relative motion may not be uniform – i.e. the fluid element may be accelerating. At any moment, however, the instantaneous velocity of the fluid element allows us to define its MCRF, although the MCRF of neighbouring elements will in general be different at that instant, and the MCRF of the fluid element will also in general be different at different times. If the fluid element is dust, however, then at any instant in the MCRF of the fluid element the individual particles possess no random motions of their own.

Generally, however, the particles within a fluid element *will* have random motions, and these will give rise to **pressure** in the fluid (c.f. motions of the molecules in an ideal gas). A fluid element may also be able to exchange energy with its neighbours via **heat conduction**, and there may be **viscous forces** present between neighbouring fluid elements. When viscous forces exist they are directed parallel to the interface between neighbouring fluid elements, and result in a **shearing** of the fluid.

A relativistic fluid element is said to be a **perfect fluid** if, in its MCRF, the fluid element has no heat conduction or viscous forces. It follows from this definition that dust is the special case of a pressure-free perfect fluid.

8.2 Definition of the energy momentum tensor

We can define the energy momentum tensor, $\mathbf{T}$, in terms of its components in some coordinate system, $\{x^1, x^2, ..., x^n\}$, for each fluid element. Thus we define $T^{\alpha\beta}$ for a fluid element to be equal to the **flux of the α component of four momentum of all gravitating matter across a surface of constant x^β.** [2]

Thus, the change, Δp^α, in the α component of the four momentum due to the flux through a surface element, ΔS_ν, at constant x^ν, is given by

$$\Delta p^\alpha = T^{\alpha\nu} \Delta S_\nu \tag{200}$$

(Note the use of the summation convention).

Mathematical aside: although we will not discuss the details of why $\mathbf{T}$ transforms as a $(2,0)$ tensor, note that this type of tensor is suggested by the definition of $\mathbf{T}$ given above. We

[2] By 'gravitating matter' we mean here all material particles, plus (from the equivalence of matter and energy) any electromagnetic fields and particle fields which may be present, although in both this course and Gravitation II we will generally consider only material particles in the form of ordinary baryonic matter

require **T** to operate on two one-forms in order to produce a scalar quantity, the flux: the first one-form selects the component, α, of the four momentum; the second one-form selects the coordinate surface, x^β = constant, across which the flux is measured. This surface is defined by a one-form. To see this note that in general a surface will be defined by an equation of the form $\phi(x^1, x^2, ..., x^n)$ = constant, for some scalar function, ϕ. (x^β = constant is simply a special case of this). The surface may, therefore, equivalently be defined by the gradient of ϕ – i.e. the one-form, $\tilde{d}\phi$. This is why we write the surface element, ΔS_ν, with a subscript, since it transforms as a one-form.

The term 'surface element' merits some further explanation for the particular case of Minkowski spacetime. Since we are considering a 4-dimensional manifold, the components of the surface element, ΔS_ν, are in fact 3-dimensional *volume* elements. Suppose we have three linearly independent small displacement vectors, $\vec{\Delta} x_{(1)}$, $\vec{\Delta} x_{(2)}$ and $\vec{\Delta} x_{(3)}$. (Here the subscript (i) is merely a label denoting which displacement vector one is referring to). Then ΔS_ν is defined to be

$$\Delta S_\nu = \epsilon_{\nu\alpha\beta\gamma} \Delta x^\alpha_{(1)} \Delta x^\beta_{(2)} \Delta x^\gamma_{(3)}$$

where $\epsilon_{\nu\alpha\beta\gamma}$ is known as the **alternating symbol** and is a $(0,4)$ tensor which takes the value 1 if $\{\nu\alpha\beta\gamma\}$ is an **even** permutation of $\{0123\}$, and -1 if it is an **odd** permutation, and zero if any two (or more) subscripts are equal. We will have no further need to discuss the alternating symbol in this course, but we introduce it here for completeness, since it helps to explain why ΔS_ν transforms as a $(0,1)$ tensor.

Suppose we choose $\vec{\Delta} x_{(1)}$, $\vec{\Delta} x_{(2)}$ and $\vec{\Delta} x_{(3)}$ to be parallel to the x^1, x^2 and x^3 axes, and write

$$\begin{aligned} \{x^\mu_{(1)}\} &= \{0, \Delta x, 0, 0\} \\ \{x^\mu_{(2)}\} &= \{0, 0, \Delta y, 0\} \\ \{x^\mu_{(3)}\} &= \{0, 0, 0, \Delta z\} \end{aligned}$$

then it follows from the properties of the alternating symbol that

$$\{\Delta S_\nu\} = \{\Delta x \Delta y \Delta z, 0, 0, 0\}$$

Similarly, if we take

$$\begin{aligned} \{x^\mu_{(1)}\} &= \{\Delta t, 0, 0, 0\} \\ \{x^\mu_{(2)}\} &= \{0, \Delta x, 0, 0\} \\ \{x^\mu_{(3)}\} &= \{0, 0, \Delta y, 0\} \end{aligned}$$

then in this case

$$\{\Delta S_\nu\} = \{0, 0, 0, \Delta t \Delta x \Delta y\}$$

from which the emergent pattern should be clear.

8.3 Components of the energy momentum tensor

Having defined $\mathbf{T}$ using equation (200) we now examine the individual components of $\mathbf{T}$ and consider what they each represent.

Recall from Section 3 that the x^0 component of the four momentum of a particle is

$$p^0 = m\gamma(v)$$

where m is the rest mass of the particle. Thus, the x^0 component of the four momentum, as measured in a particular Lorentz frame, is the total energy of the particle. (Remember that in relativistic physics energy and momentum are equivalent, so that all four components of the four momentum together specify the total energy and momentum of a particle, with the relative amount of each dependent the Lorentz frame in which they are measured). The total energy is the sum of the rest mass energy (i.e. mc^2 with $c = 1$) and kinetic energy, thus

$$\text{Total energy} \quad = \quad m + m(\gamma - 1)$$

where the second term reduces to the familiar Newtonian expression for kinetic energy in the limit of $v << 1$

Consider first the components of $\mathbf{T}$ in the MCRF of a fluid element. We begin with the '$(0,0)$' component. Taking small displacement vectors parallel to the spatial axes we obtain $\{\Delta S_\nu\} = \{\Delta x \Delta y \Delta z, 0, 0, 0\}$. (See the mathematical aside above for details). Thus, from equation (200) it follows that

$$\Delta p^0 = T^{00} \Delta x \Delta y \Delta z \tag{201}$$

Hence T^{00} is the **energy density** of the fluid element. Similarly

$$\Delta p^i = T^{i0} \Delta x \Delta y \Delta z \quad (i = 1, 2, 3) \tag{202}$$

from which we see that T^{i0} is the i^{th} component of the **momentum density** of the fluid element.

Consider now T^{0i}, for $i = 1, 2, 3$. Taking T^{01} first, by a similar argument to that given above we choose small displacement vectors parallel to the x^0, x^2 and x^3 axes, to obtain $\{\Delta S_\nu\} = \{0, \Delta y \Delta z \Delta t, 0, 0\}$, from which it follows that

$$\Delta p^0 = T^{01} \Delta y \Delta z \Delta t \tag{203}$$

But $\Delta y \Delta z$ is the area of the face of our volume element perpendicular to the x^1 axis. Thus, T^{01} denotes the **energy flux** (i.e. energy per unit area per unit time) from the fluid element in the x^1 direction. Clearly the obvious corresponding interpretations follow for T^{02} and T^{03}.

Finally, consider T^{ij}, for $i, j = 1, 2, 3$. Taking T^{i1} as our illustration, we again choose small displacement vectors parallel to the x^0, x^2 and x^3 axes, to obtain the same components of ΔS_ν as above, so that

$$\Delta p^i = T^{i1} \Delta y \Delta z \Delta t \quad (i = 1, 2, 3) \tag{204}$$

Thus T^{i1} is the flux of the i^{th} component of the momentum in the x^1 direction, with the corresponding meaning for T^{i2} and T^{i3}.

The transfer of momentum across the interface between neighbouring fluid elements indicates that the elements exert a force on each other. If the forces are directed *perpendicular* to the interface between neighbouring fluid elements, then we see that such forces are represented by the *diagonal* T^{ij} components, i.e. with $i = j$. If, on the other hand, the forces are directed *parallel* to the interface between neighbouring fluid elements, then they are represented by the *off-diagonal* T^{ij} components, i.e. with $i \neq j$.

8.4 Symmetry of the energy momentum tensor

An important property of the energy momentum tensor which we will state here, but not prove, is that **T** is **symmetric** – i.e. in any coordinate system $T^{\alpha\beta} = T^{\beta\alpha}$, $(\alpha, \beta = 0, 1, 2, 3)$. (See Tutorial Sheet 3 for a proof that the symmetry of a tensor must hold under any coordinate transformation). To see a proof that the energy momentum tensor is symmetric, in the MCRF of a fluid element and hence in any frame, refer to e.g. Schutz, p. 102 et. seq.

8.5 Energy momentum tensor for a perfect fluid

8.5.1 Components of T in the MCRF for dust

In this case the energy momentum tensor takes a very simple form. Since the particles in the fluid element are at rest, there is no momentum transfer. (For a general fluid, even if the particles are at rest there can be a flux of energy and momentum through heat conduction, but not for dust, which is a perfect fluid). Also there is no momentum flux, which means that $T^{ij} = 0$, $(i, j = 1, 2, 3)$. In fact the only non-zero component is $T^{00} = \rho$, the energy density of the fluid element.

8.5.2 Components of T in the MCRF for a general perfect fluid

This case is only slightly less straightforward than that of dust. Again the T^{00} component is equal to the energy density, ρ. Since there is no bulk motion of the fluid element and there is no heat conduction for a perfect fluid, the energy flux $T^{0i} = 0$ for $i = 1, 2, 3$. Moreover, from the symmetry of **T** we also have that the momentum density, $T^{i0} = 0$, for $i = 1, 2, 3$. For the spatial components, $T^{ij} = 0$ if $i \neq j$, since these terms correspond to viscous forces parallel to the interface between fluid elements and these forces are zero for a perfect fluid. Thus T^{ij} is a diagonal matrix. But T^{ij} must be diagonal in all reference frames – e.g. under all possible rotations. This is possible only if T^{ij} is a scalar multiple of the identity matrix, i.e. $T^{11} = T^{22} = T^{33}$.

Thus, T^{ii} is the flux of the i^{th} component of momentum in the x^i direction, perpendicular to the fluid element interface. Equivalently, it is the *force per unit area*, perpendicular to the interface. This is just the **pressure**, P, exerted by the random motions of the particles in the fluid element. Hence we can write **T** as

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{205}$$

8.5.3 Components of T in a general Lorentz frame

Consider now the components of **T** in an arbitrary Lorentz frame. We again begin with the case of dust. If we suppose that all particles in the fluid element have equal rest mass (energy), m, then we can write the energy density as

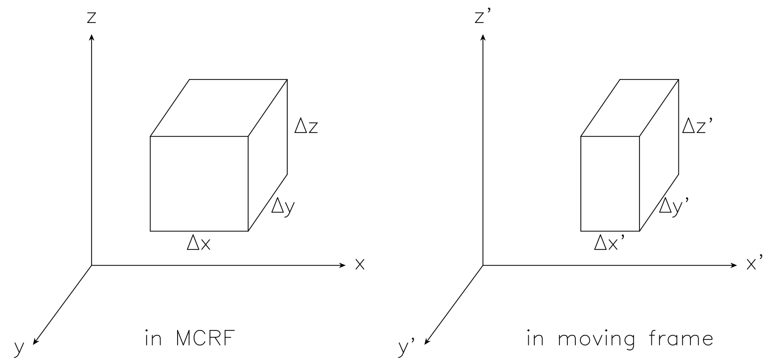
$$\rho = mn \tag{206}$$

where n is the *number density* of particles in the fluid element.[3] Suppose , in the MCRF, the fluid element is a cuboid with spatial dimensions Δx, Δy and Δz parallel to the x^1, x^2 and x^3 axes (see Figure 15). Then the number of particles in the fluid element in the MCRF is simply

$$N = n\Delta x\Delta y\Delta z \tag{207}$$

[3]Even if the particles have different rest masses, we can still write an equation of the same form as (206), replacing m with a weighted average over the fluid particles

Figure 15



Now suppose that we view the fluid element in a primed Lorentz frame which is moving parallel to the x^1 axis with velocity, v. Assuming fluid particles are neither created nor destroyed, then N will be constant but Δx will be Lorentz contracted. (Δy and Δz are unchanged). Thus, the number density measured in the primed frame will be

$$n' = n\gamma(v) \tag{208}$$

but the *energy* of the particles measured in the primed frame will also be increased by a factor of γ. Thus the energy density measured in the primed frame will be

$$\rho' = T'^{00} = \frac{\rho}{1-(v)^2} \tag{209}$$

It is straightforward, though tedious, to show that the same expression holds when the velocity, v, is not parallel to any of the coordinate axes and has arbitrary 3-velocity components $\{v^1, v^2, v^3\}$.

A similar argument then shows that the energy *flux* across the x'^j surface in the primed frame is given by

$$T'^{0j} = \frac{\rho v^j}{1-(v)^2} \tag{210}$$

and the i^{th} component of momentum flux across the x'^j surface by

$$T'^{ij} = \frac{\rho v^i v^j}{1-(v)^2} \tag{211}$$

We can collect together all of these terms into the general expression

$$T'^{\alpha\beta} = \rho u'^{\alpha} u'^{\beta} \tag{212}$$

where $\vec{u}' = \{u'^{\alpha}\}$ is the *four* velocity of the fluid element in the primed frame. (Compare with equations (52) - (55) to see this).

For a perfect fluid, with non-zero pressure, equation (212) further generalises to

$$T^{\alpha\beta} \;=\; (p+P)u^{\alpha}u^{\beta} \;+\; P\eta^{\alpha\beta} \tag{213}$$

(where for clarity we have dropped the primes). Here $\eta^{\alpha\beta}$ are the contravariant components of the metric tensor for Minkowski spacetime (see Section 3). It is easy to see that when $v = 0$, equation (213) reduces to equation (205).

8.6 Conservation of Energy and Momentum

Conservation of momentum and energy requires that

$$T^{\mu\nu}_{\;\;,\nu} = 0 \tag{214}$$

To see this, consider again the 3-d spatial volume of Figure 15, with dimensions Δx, Δy and Δz, and consider the **energy** passing into and out of the volume in time $\Delta t = \Delta x^0$.

The energy through the 2-d surface ΔA_x, perpendicular to the x^1 axis, in time Δt is given by

$$\Delta E_{\rm in} = \Delta p^0_{\rm in} = T^{01}(t,x,y,z)\Delta A_x \Delta t \tag{215}$$

The energy out of the 2-d surface $\Delta A_{x+\Delta x}$, perpendicular to the $x^1 \equiv x$ axis, in time Δt is similarly given by

$$\Delta E_{\rm out} = \Delta p^0_{\rm out} = T^{01}(t,x+\Delta x,y,z)\Delta A_x \Delta t \tag{216}$$

Thus the net energy into the volume in the x^1 direction is

$$\Delta E(x) = T^{01}(t,x,y,z)\Delta A_x \Delta t - T^{01}(t,x+\Delta x,y,z)\Delta A_x \Delta t \tag{217}$$

which can be approximated as

$$\Delta E(x) = -\frac{\partial T^{01}}{\partial x^1}\Delta x \Delta A_x \Delta t = -\frac{\partial T^{01}}{\partial x^1}\Delta V \Delta t \tag{218}$$

Similarly the net energy into the volume element ΔV in time Δt in the $x^2 \equiv y$ direction and $x^3 \equiv z$ direction respectively is given by

$$\Delta E(y) = -\frac{\partial T^{02}}{\partial x^2}\Delta V \Delta t \tag{219}$$

and

$$\Delta E(z) = -\frac{\partial T^{03}}{\partial x^3}\Delta V \Delta t \tag{220}$$

The total increase in energy inside the volume ΔV in time Δt is thus given by

$$\Delta E = -(\frac{\partial T^{01}}{\partial x^1} + \frac{\partial T^{02}}{\partial x^2} + \frac{\partial T^{03}}{\partial x^3})\Delta V \Delta t \tag{221}$$

On the other hand the increase in energy within ΔV is also simply

$$\Delta E = T^{00}(t + \Delta t, x, y, z)\Delta V - T^{00}(t, x, y, z)\Delta V \tag{222}$$

which is approximately

$$\frac{\partial T^{00}}{\partial x^0}\Delta V \Delta t \tag{223}$$

Equating (221) with (223) yields

$$\frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{01}}{\partial x^1} + \frac{\partial T^{02}}{\partial x^2} + \frac{\partial T^{03}}{\partial x^3} = 0 \tag{224}$$

This establishes equation (214) for the case where $\mu = 0$. A similar procedure, considering the change in momentum flux across ΔV, will yield the remaining equations for $\mu = 1, 2, 3$. Thus, the **divergence** of the energy momentum tensor is equal to zero.

8.7 Extending to GR

In Section 1 we introduced the strong principle of equivalence which stated that, in a local inertial frame, all physical phenomena are in agreement with special relativity. In the light of our study of tensors in this course, we can write down an immediate consequence of the strong principle of equivalence as follows

Any physical law which can be expressed as a tensor equation in SR has exactly the same form in a local inertial frame of a curved spacetime

This statement holds since, in the LIF, physics – and hence the form of physical laws – is indistinguishable from the physics of SR. This is a very important result because it allows us to generalise the form of physical laws which are valid in SR to the case of GR, with semi-colons (denoting covariant derivatives) replacing commas (denoting partial derivatives) where appropriate.

How is this extension justified? From the principle of covariance a tensorial description of physical laws must be equally valid in any reference frame. Thus, if a tensor equation holds in one frame it must hold in any frame. In particular, a tensor equation derived in a LIF (i.e. assuming SR) remains valid in an arbitrary reference frame (i.e. assuming GR).

Hence, the energy momentum tensor for a perfect fluid in GR takes the form

$$T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + Pg^{\mu\nu} \tag{225}$$

where $g^{\mu\nu}$ denotes the contravariant metric tensor for a general curved spacetime (which of course reduces locally to $\eta^{\mu\nu}$).

We can extend to GR in this way the result of equation (214), on the conservation of energy and momentum. Thus, for a fluid element in a general curved spacetime

$$T^{\mu\nu}_{;\nu} = 0 \tag{226}$$

If this were *not* the case – i.e. if there existed some point, P, at which $T^{\mu\nu}_{;\nu} \neq 0$ – then we could construct a LIF at P (e.g. by changing to geodesic coordinates) in which all Christoffel symbols are zero. In this new frame covariant derivatives reduce to partial derivatives, implying that $T^{\mu\nu}_{,\nu} \neq 0$, which contradicts equation (214).

The general technique of using the principles of covariance and equivalence to extend the validity of tensor equations from SR to GR, usually by evaluating their components in the LIF where Christoffel symbols vanish, is a very powerful one and is commonly met in the relativity literature. It is sometimes referred to informally as the 'comma goes to semi colon rule'.