7 Geodesics

Earlier we stated that material particles not acted on by forces other than gravitational forces have worldlines that are geodesics. Similarly photons also follow geodesics. Yet we haven't yet given a complete definition of a geodesic.

One can adopt one of two approaches. Either one can define a geodesic as an extremal path between two events, in the sense that the proper time along the path joining the two events is an extremum.

We shall take a different approach here, and define the geodesic in terms of parallel transport. First, recall from Section 4, equation (93), that we defined a *curve* as a parametrised path, i.e.

$$x^\mu = x^\mu(\eta) \tag{174}$$

where η is a parameter and μ runs over the four indices. We can define a tangent vector to this path with components

$$T^\mu = \frac{dx^\mu}{d\eta} \tag{175}$$

We now define a *geodesic as a curve along which the tangent vector to the curve is parallel-transported*. We do not insist that the length of the transported tangent vector should have the same length as the tangent vector at the new point (note that this would in any case only have a meaning for Riemannian spaces, for which a metric is defined). We demand only that it be parallel. (The magnitude of a tangent vector along a curve will depend on the parameterisation of the curve. A change in parametrisation gives a different curve, but the same set of points. A change in parametrisation will change the magnitude of the tangent vector).

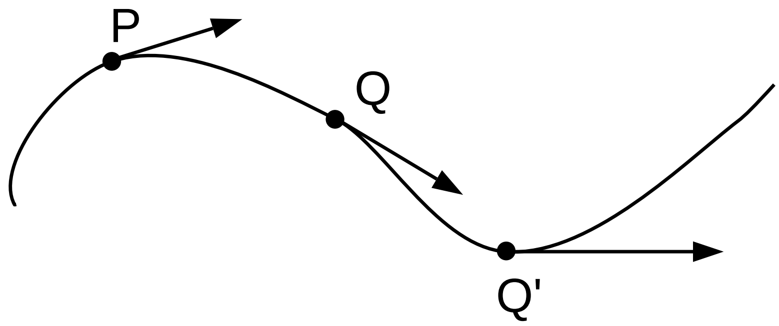
When we parallel transport a tangent vector at an arbitrary point P to a neighbouring point Q along the geodesic curve, then it is in the same direction as the tangent vector at Q. (See Figure 14).

Thus if the curve $x^\mu = x^\mu(\eta)$ is a geodesic, when the tangent vector, $\vec{T}_P$, is parallel transported from point P (with parameter value η_0, say) to point Q (with parameter value η) and tangent vector $\vec{T}_Q$, it follows that

$$\vec{T}_P \rightarrow f(\eta)\,\vec{T}_Q \tag{176}$$

The scalar function, $f(\eta)$, depends on the parameterisation of the curve.

Figure 14



Consider now a point, Q', in the neighbourhood of Q with parameter $\eta + d\eta$. Suppose we parallel transport $\vec{T}_P$ directly to Q'. This will yield

$$\vec{T}_P \;\rightarrow\; f(\eta + d\eta)\,\vec{T}_{Q'} \tag{177}$$

which must be the same as we obtain by first parallel transporting $\vec{T}_P$ to Q and then parallel transporting $f(\eta)\vec{T}_Q$ to Q'. Consider the second step, from Q to Q'. Writing the parallel-transported vector at Q' in component form, and using equations (138) and (142), we obtain

$$f(\eta + d\eta)\,T^\mu(\eta + d\eta) \;=\; f(\eta)T^\mu(\eta) - f(\eta)\Gamma^\mu_{\alpha\beta}T^\alpha(\eta)T^\beta(\eta)d\eta \tag{178}$$

where here we have written the term corresponding to dx^k in equation (142) as $T^\beta(\eta)d\eta$.

Expanding to first order in $d\eta$ we obtain the **geodesic equation**

$$f(\eta)\frac{dT^\mu}{d\eta} + T^\mu\frac{df}{d\eta} + f(\eta)\Gamma^\mu_{\alpha\beta}T^\alpha(\eta)T^\beta(\eta) \;=\; 0 \tag{179}$$

In this equation the choice of parameter η was arbitrary. The equation can be reduced to a somewhat simpler form by choosing a new parameter. A tedious, though fairly straightforward, calculation shows that a change in parameter to λ defined by

$$\lambda \;=\; -C_0\int f(\eta)^{-1}d\eta \;+\; \lambda_0 \tag{180}$$

where C_0 and λ_0 are constants, gives a new geodesic equation

$$\frac{ds^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}s^\alpha s^\beta = 0 \tag{181}$$

where

$$s^\mu = \frac{dx^\mu}{d\lambda}$$

λ is called an **affine parameter**. The above equation can evidently be written as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{182}$$

Notice that any linear transformation of an affine parameter also gives affine parameter.

7.1 Geodesics of material particles

Consider the worldline of a material particle. In a given coordinate system the the worldline may be written with the proper time, τ, along the worldline as the parameter, i.e.

$$x^\mu = x^\mu(\tau) \tag{183}$$

The four velocity of the particle is given by

$$v^\mu = \frac{dx^\mu}{d\tau} \tag{184}$$

and is the tangent vector to the worldline. In this case the function, f, in equation (176) is simply a constant and equal to either $+1$ or -1. We can take it to be $+1$, and it then is clear from equation (180) that τ is an affine parameter. Thus, we obtain the geodesic equation (182) with τ as parameter, i.e.

$$\frac{dv^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta = 0 \tag{185}$$

or alternatively

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{186}$$

One can show (see Examples Sheet 5) that

$$g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{187}$$

is constant along a geodesic, i.e.

$$\frac{d}{d\tau}(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}) = 0 \tag{188}$$

7.2 Geodesics of photons

For photons, the proper time τ cannot be used to parametrize the worldlines, since $d\tau$ is zero. If we use an arbitrary affine parameter λ the null geodesics will be described by

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{189}$$

and since it is a null geodesic, necessarily

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{190}$$

7.3 Examples

Example 1: Consider the almost trivial example

$$ds^2 = -dt^2 + dx^2 = -d\tau^2 \tag{191}$$

For a material particle we have the geodesic equations

$$\frac{d^2x}{d\tau^2} = 0$$

$$\frac{d^2t}{d\tau^2} = 0$$

which have solution

$$x = A\tau + B$$

$$t = C\tau + D$$

where A, B, C and D are constants of integration. Evidently the velocity of the particle is given by A/C. Furthermore, since

$$-(\frac{dx}{d\tau})^2 + (\frac{dt}{d\tau})^2 = 1$$

we see that A and C are not independent, but satisfy $C^2 - A^2 = 1$.

For a photon we cannot use the proper time τ. Introducing an arbitrary affine parameter, λ, we have

$$\frac{d^2x}{d\lambda^2} = 0$$

$$\frac{d^2t}{d\lambda^2} = 0$$

giving

$$x = A\lambda + B, t = C\lambda + D$$

However we know that

$$(\frac{dx}{d\lambda})^2 - (\frac{dt}{d\lambda})^2 = 0 \tag{192}$$

which implies that $A^2 = C^2$. Thus null geodesics have the form

$$x = A\lambda + B, \quad t = A\lambda + D$$

or

$$x = -A\lambda + B, \quad t = A\lambda + D$$

Example 2: Consider now the metric

$$dl^2 = dr^2 + r^2 d\theta^2 \tag{193}$$

which obviously corresponds to 2-d Euclidean space in polar coordinates.

The distance, l, along a curve is an affine parameter since

$$g_{ij} \frac{dx^i}{dl} \frac{dx^j}{dl} = (\frac{dr}{dl})^2 + r^2 \, (\frac{d\theta}{dl})^2 = 1 \tag{194}$$

Thus the geodesic equations are

$$\frac{d^2 r}{dl^2} + \Gamma^r_{\theta\theta} \frac{d\theta}{dl} \frac{d\theta}{dl} + \Gamma^r_{\theta r} \frac{d\theta}{dl} \frac{dr}{dl} + \Gamma^r_{r\theta} \frac{dr}{dl} \frac{d\theta}{dl} + \Gamma^r_{rr} \frac{dr}{dl} \frac{dr}{dl} = 0 \tag{195}$$

and

$$\frac{d^2 \theta}{dl^2} + \Gamma^\theta_{\theta\theta} \frac{d\theta}{dl} \frac{d\theta}{dl} + \Gamma^\theta_{\theta r} \frac{d\theta}{dl} \frac{dr}{dl} + \Gamma^\theta_{r\theta} \frac{dr}{dl} \frac{d\theta}{dl} + \Gamma^\theta_{rr} \frac{dr}{dl} \frac{dr}{dl} = 0 \tag{196}$$

In equations (143) - (146) we derived the values of the Christoffel symbols in polar coordinates. Substituting these values, the geodesic equations simplify to

$$\frac{d^2 r}{dl^2} - r(\frac{d\theta}{dl})^2 = 0 \tag{197}$$

and

$$\frac{d^2 \theta}{dl^2} + \frac{2}{r}(\frac{d\theta}{dl})^2 = 0 \tag{198}$$

respectively.

One obvious set of solutions to these equations is $r = l$, $\theta =$ constant. In fact any straight line will satisfy these equations.

7.4 Geodesics as extremal paths

We chose to define a geodesic by using the notion of parallel transport. This did not in fact depend on the notion of a metric, but only of an affine space. As we have already mentioned, in a metric space one could also define a geodesic as an extremal path in the sense that, along the geodesic between two events E_1, and E_2

$$\delta \int_{E_1}^{E_2} d\tau = 0 \tag{199}$$

We shall not prove it here, but equation (199) also yields the geodesic equation (182).