5 Spacetime and the metric

5.1 The spacetime metric

Spacetime is a 4 dimensional manifold. The points of this manifold are called **events**. We can also define a *distance*, or interval, between neighbouring events – i.e. spacetime is a *Riemannian* manifold. If the interval between these events in **timelike**, we define the distance between them as the **proper time** recorded by a particle on whose worldline the events lie.

Suppose now that the separation between the events is **spacelike**. If the separation between the events is sufficiently small, one can choose as one's coordinate system a local inertial frame in which the 'distance' between the events is simply be given by (with $c = 1$).

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{118}$$

However, because spacetime is curved, no coordinate system can be constructed in which the distance between any two arbitrary events can always be expressed by equation (118). Only a **local** Lorentz frame can be found. The line element given by equation (118) cannot be valid throughout spacetime if the spacetime is curved.

(In exactly the same way, no coordinate system can be found in which the line element for the sphere embedded in 3D Euclidean space reduces everywhere to the Cartesian form $dl^2 = dx^2 + dy^2$)

Suppose a coordinate system has been set up in spacetime. Each event, P, is provided with 4 coordinate values, say $\{x^0, x^1, x^2, x^3\}$. These coordinates can be quite general, and are not necessarily assumed to represent time and spatial coordinates. A neighbouring event, Q, has coordinates $\{x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3\}$. We shall write the (invariant) distance between these events as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{119}$$

$g_{\mu\nu}$ is called the **metric tensor** (stricly speaking $\mathbf{g}$, in coordinate-free notation, is the metric tensor and $g_{\mu\nu}$ is just the metric tensor components, but this distinction is rarely made in practice).

To say that spacetime is *locally Minkowskian* (sometimes loosely stated as 'locally flat') means that there exists a coordinate transformation that reduces equation (119) to the form of equation (118) in the neighbourhood of event P. The assumption that spacetime locally reduces to Minkowski form is tantamount to assuming that it is possible to transform to a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu}$ at event P. (We shall see that in fact we can also find a transformation such that $g_{\mu\nu,\alpha} = 0$ at P. Such a coordinate system is called a geodesic coordinate system at P.)

If ds^2 is not **positive definite**, it is called a *pseudo-Riemannian* space. (A metric is positive definite if ds^2 is always greater or equal to zero, and equal to zero only when $dx^\mu = 0$).

We can choose $g_{\mu\nu}$ to be symmetric in μ and ν, which means that it has 10 independent components. Remember that in general $g_{\mu\nu}$ are functions of the coordinates (x^0, x^1, x^2, x^3).

5.2 Transformation law for the metric

At any event, P, the value of $g_{\mu\nu}$ will depend on the coordinates used. It is easy to see that the metric must, in fact, transform as a $(0,2)$ tensor, since ds^2 is invariant, and $dx^\mu dx^\nu$ transforms as a $(2,0)$ tensor. Thus we have the transformation law

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \tag{120}$$

which generalises, and justifies, equation (47).

5.3 Role of the metric tensor in GR

The metric tensor, $g_{\mu\nu}$, describes the geometric properties of spacetime. GR, the theory of spacetime, must describe how the intrinsic properties of spacetime are affected and determined by the presence and movement of gravitating matter.

How will $g_{\mu\nu}$ enter the field equations of GR? There are several clues. We saw in Section 1 that the equivalence of gravitational and inertial mass could be explained by insisting that test particles move along geodesics in spacetime. We further argued that the acceleration of the deviation of test particles (geodesic deviation, ξ) was determined by the 'curvature' of spacetime. Hence one should expect second order derivatives of $g_{\mu\nu}$ to play a key role since these are necessary to express the curvature. So if the field equations are to be covariant, we must find a tensorial quantity involving second order derivatives of $g_{\mu\nu}$ that describes the geometry of spacetime and that can be related to the matter and energy content of the Universe (which we will describe by another tensor, the *energy-momentum* tensor). There should evidently be further constraints on the theory.

- In the non-relativistic limit Newtonian gravity should be regained.
- The theory should also be consistent with SR in the limit of low matter density.

These constraints mean that the theory should be consistent with energy and momentum conservation.

5.4 Contravariant components of the metric tensor

The components g_{ij} transform as covariant components. We can also define contravariant components of the metric tensor g^{jk} by requiring

$$g_{ik}g^{jk} = \delta_i^j \tag{121}$$

g^{jk} defined in this way must be unique, since g_{ij} is nonsingular. g^{jk} transforms as a $(2, 0)$ tensor, since g_{ik} transforms as a $(0, 2)$ tensor and δ_i^j as a $(1, 1)$ tensor.

Exercise: The line element in 2-D Euclidean space is given by

$$dl^2 = dx^2 + dy^2$$

Show that

$$g^{xx} \;=\; g^{yy} \;=\; 1, \;\; g^{xy} \;=\; g^{yx} \;=\; 0$$

Write down the line element in terms of the coordinates $\{u, v\}$ where $u = ax + by$ and $v = cx + dy$, where a, b, c, d are constants. Using the fact that g_{ij} and g^{ij} respectively transform covariantly and contravariantly, determine them in the coordinate system $\{u, v\}$.

Exercise: Write down the contravariant and covariant components of the metric tensor for $\mathbf{E^2}$ in polar coordinates, i.e. $\{r, \theta\}$, defined by $x = r\cos\theta$, $y = r\sin\theta$.

5.5 Raising and lowering indices and contraction

Given any contravariant vector A^i it is possible to define, via the metric tensor, an associated one-form, which we denote as A_i and which is defined by

$$A_i = g_{ik}A^k \tag{122}$$

This operation is often called *lowering the index*.

Similarly by using g^{ij} we can raise the index of a covariant quantity B_i to obtain a contravariant quantity B^i, viz.

$$B^i = g^{ij}B_j \tag{123}$$

An example of raising the index would be to obtain the components of the **vector gradient**, $\vec{d\phi}$, from the one-form gradient, $\tilde{d\phi}$, defined in equation (111). Thus

$$(\vec{d\phi})^i = g^{ij}(\tilde{d\phi})_j \tag{124}$$

In elementary courses on vector calculus, the gradient is usually introduced as a vector: equation (123) establishes the relationship between the gradient as a one-form (as we have introduced it in this course) and as a vector. Note that from equation (123) it follows immediately that for Euclidean space with a Cartesian basis (for which the metric tensor is equal to the identity matrix) the components of a contravariant vector and its associated one-form are, in fact, identical. Thus, we see that in this particular case no distinction need be made between the one-form and vector description of the gradient (but see Problem Sheet 2 for an example of where, even for Euclidean space, the distinction between one-forms and vectors is non-trivial).

The role of the metric in connecting one-forms and vectors is crucial. We can see this qualitatively by considering again the topographic map of Figure 11. Our intuitutive idea of the vector gradient involves an arrow pointing in the direction in which the contours of the map are changing most rapidly. In order to define this, we need a measure of how many contours are crossed *per unit length* in a particular arrow direction. Thus, we see that the vector gradient is larger in magnitude at position (1) than at positions (2) or (3). However, the crucial point in this picture is the phrase *per unit length* – i.e. in order to define a vector gradient we first need to define the notion of what we mean by *length*: in other words we need a *metric*.

The process of raising or lowering indices can be carried out with tensors of any rank and type. For example

$$D^{ijk..}_{lm} = g_{lp} g_{mq} D^{ijkpq} \tag{125}$$

Some care must be taken in positioning the indices. The dots have been placed here to indicate the indices over which contraction has taken place, although in general we shall omit the dots and just write D^{ijk}_{lm}. Note that $D^{..ijk}_{lm}$ defined by

$$D^{..ijk}_{lm} = g_{lp} g_{mq} D^{pqijk} \tag{126}$$

is not the same as $D^{ijk..}_{lm}$ unless D^{ijkpq} possesses some symmetry.

Exercise: $T^{\mu\nu}$ is symmetric. Show that $T_{\mu}{}^{\nu} = T^{\nu}{}_{\mu}$.

The *magnitude* of a vector A^i is $g_{ij} A^i A^j$, which is of course invariant, since g_{ij} is a $(0,2)$ tensor and A^i and A^j are both $(1,0)$ tensors. Notice

$$g_{ij} A^i A^j = A_j A^j = g^{ij} A_i A_j \tag{127}$$

$g_{ij} A^i B^j$ may be regarded as the scalar product of two vectors.