4 Manifolds

To deal with curved spacetime – and to allow us to define geometrical objects such as vectors and tensors in a completely general way – we have to introduce the notion of a *manifold*. A manifold is essentially a continuous space which is locally flat. More generally we can regard a manifold as any set which can be continuously parametrised: the number of independent parameters is the *dimension* of the manifold, and the parameters themselves are the *coordinates* of the manifold. A *differentiable manifold* is one which is both continuous and differentiable. This means that we can define a scalar function (or *scalar field*) – ϕ, say – at each point of the manifold, and that ϕ is differentiable. We describe this more explicitly, in terms of coordinates, in the next section.

Examples of differentiable manifolds are:

1. the two dimensional plane, $\mathbf{E^2}$
2. the two dimensional sphere, $\mathbf{S^2}$
3. a two dimensional torus (e.g. surface of a doughnut)

In this course we will be concerned with a particular class of differentiable manifolds known as *Riemannian* manifolds. A Riemannian manifold is a differentiable manifold on which a *distance*, or *metric*, has been defined.

Evidently $\mathbf{E^2}$ has a natural distance defined on it. If we take a Cartesian coordinate system $\{x, y\}$ the distance, dl, between two neighbouring points, P, with coordinates (x, y), and Q, with coordinates $(x + dx, y + dy)$, is given by

$$dl^2 = dx^2 + dy^2 \tag{78}$$

dl^2 is often called the *line element*. Note that the distance between P and Q is defined in a coordinate-free way – i.e. it exists independently of one's choice of coordinate system, although in equation (78) a Cartesian coordinate system has been chosen to represent it. If we use instead a polar coordinate system, $\{r, \theta\}$, where

$$r = (x^2 + y^2)^{\frac{1}{2}} \tag{79}$$

and

$$\theta = \text{atan}\frac{y}{x} \tag{80}$$

then the distance between P and Q may be written as

$$dl^2 = dr^2 + r^2 d\theta^2 \tag{81}$$

The Minkowski spacetime of special relativity is another example of a Riemannian manifold, with metric defined via equation (33).

If one imagines a curved manifold embedded in a higher dimensional Euclidean space (e.g. a 2-D sphere embedded in $\mathbf{E^3}$) then there is a natural metric which we can adopt, which is just the Euclidean distance function of the higher-dimensional space in which the manifold is embedded. Thus examples (ii) and (iii) above are usually seen as embedded in $\mathbf{E^3}$, in which case it is straightforward to derive the expression for the line element.

Exercise: Show that the line element for the surface of a sphere of radius, R, embedded in 3-D Euclidean space is given in spherical polar coordinates by

$$dl^2 = R^2(d\theta^2 + sin^2\theta d\phi^2) \tag{82}$$

Exercise: Find a coordinate system for a torus, and write down the line element in this coordinate system.

4.1 Differentiable manifolds

Consider a point, P, in a Riemannian manifold, to which a set of coordinates has been assigned.

> **Mathematical Aside:** The formal mathematics of defining coordinates for P need not concern us in this lecture course, but the interested reader can find useful discussions in any introductory textbook on differential geometry. Loosely speaking, it involves covering the points of the manifold by a collection of *open sets*, $\mathbf{U^i}$, each of which is mapped onto $\mathbf{R^n}$ by a one-to-one mapping, ϕ^i. The pair $(\mathbf{U^i}, \phi^i)$ is called a *chart*, and the collection of charts an *atlas*. One can think of each chart as defining a different coordinate system.

Suppose point P belongs to $\mathbf{U}$ and $\mathbf{U}'$. From chart $(\mathbf{U}, \phi)$, P has coordinates $\{x^1, x^2, ..., x^n\}$ and from $(\mathbf{U}', \phi')$ it has coordinates $\{x'^1, x'^2, ..., x'^n\}$. There will be a functional relationship between the two sets of coordinates; i.e. we can write

$$x'^j = f^j(x^1, x^2, ..., x^n) \tag{83}$$

for each $j = 1, 2, ..., n$. In somewhat loose notation we can also, for expedience, write this as

$$x'^j = x'^j(x^1, x^2, ..., x^n) \tag{84}$$

If the all partial derivatives of $f^j(x^1, x^2, ..., x^n)$ exist for all orders then we call the manifold differentiable. With this notion of a differentiable manifold, a great deal of geometric structure can be defined, including **functions**, **tangent vectors**, **one-forms** and **tensors**.

4.2 Functions on a manifold

One can define a function, f, on a manifold, $\mathbf{M}$. At any point, P, of the manifold the function takes a real value

$$f : \mathbf{M} \to \mathbf{R} \tag{85}$$

In a particular coordinate representation, P has coordinates $\{x^1, x^2, ..., x^n\}$. We may then write simply

$$f_P = f(x^1, x^2, ..., x^n) \tag{86}$$

In another, primed, coordinate coordinate system P has coordinates $\{x'^1, x'^2, ..., x'^n\}$. Thus we may write

$$\begin{aligned} f_P &= f(x^1, x^2, ..., x^n) \\ &= f\left(x^1(x'^1, x'^2, ..., x'^n), x^2(x'^1, x'^2, ..., x'^n), ..., x^n(x'^1, x'^2, ..., x'^n)\right) \\ &= f'(x'^1, x'^2, ..., x'^n) \end{aligned} \tag{87}$$

f is called a *scalar* function; this means that its numerical value at each point of the manifold is the same real number, no matter which coordinate representation is used.

4.3 Vectors and one-forms

The intuitive picture of a vector which we have learned in elementary maths and physics courses is based on the simple idea of an arrow representing a *displacement* between two points in space. In section 2 we introduced the idea that a vector, $\vec{a}$, exists independently of our choice of coordinate system, but the *components* of $\vec{a}$ take different values in different coordinate systems, and we can define a transformation law for the components of the vector. Consider, for example, the displacement vector, $\vec{\Delta x}$, with components Δx^μ and $\Delta x'^\mu$ in an unprimed and primed coordinate system respectively. Recall from equation (43) that

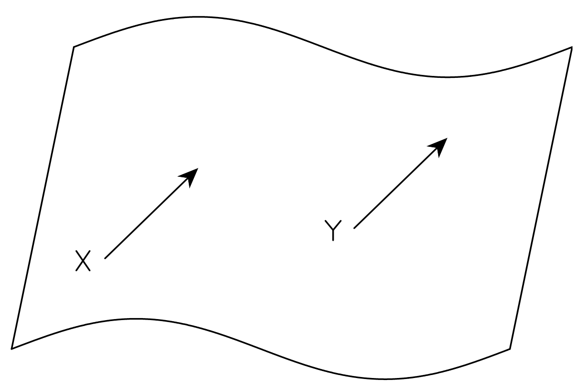
$$\Delta x'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} \Delta x^\alpha \tag{88}$$

Consider now two displacement vectors, $\Delta\vec{x}$ and $\Delta\vec{y}$. How can we decide if $\Delta\vec{x}$ and $\Delta\vec{y}$ are equal when – as shown in Figure 7 – they are defined at different points on our manifold? For vectors in $\mathbf{E^3}$ with Cartesian coordinates, for example, we can simply 'translate' $\Delta\vec{y}$ to X and compare the components of $\Delta\vec{y}$ with those of $\Delta\vec{x}$. We can do this for *any* point, Y, and in *any* coordinate system, however, **only** if the transformation law for vectors is the same at every point of the manifold. This is indeed the case for Cartesian coordinates in $\mathbf{E^3}$, but will **not** be true for a general curved manifold. This is because for a curved manifold the coefficients of the transformation law in equation (88) are in general functions of position, i.e.

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \frac{\partial x'^{\mu}(x^1, x^2, ..., x^n)}{\partial x^{\alpha}} \tag{89}$$

In other words, the transformation law between the primed and unprimed coordinate systems is in general *different* at different points of the manifold. Thus, it is *not* enough to define the components of a vector; we also need to specify the point of the manifold at which the vector (and its components) are defined.

Figure 7



The fact that the transformation law coefficients of equation (88) are in general functions of position also means that we have no 'universal' set of coordinate basis vectors on a curved manifold, as is the case for Euclidean space. (In fact, this is precisely why it is not enough simply to define the components of a vector – with respect to some fiducial set of basis vectors – since these vectors will *not* in general form a basis at every point of the manifold). There is, however, a means of defining a natural set of basis vectors for each point of the manifold which allows us to develop a more general picture of what we mean by a vector – and one which is equally valid in a curved spacetime.

4.3.1 Tangent vectors

Suppose we have a scalar function, ϕ, defined at a point, P, of a Riemannian manifold, where P has coordinates $\{x^1, x^2, ..., x^n\}$ in some coordinate system. Since our manifold is differentiable we can evaluate the derivative of ϕ with respect to each of the coordinates, x^i, for $i = 1, ..., n$. In fact, since ϕ is completely arbitrary, we can think of the derivatives as a set of n 'operators', denoted by

$$\frac{\partial}{\partial x^i}$$

These operators act on any scalar function, ϕ, and yield the rate of change of the function with respect to the x^i.

We can now define a **tangent vector** at point, P, as a linear operator of the form

$$a^\mu \frac{\partial}{\partial x^\mu} \equiv a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + ... + a^n \frac{\partial}{\partial x^n} \tag{90}$$

(Note the use of the summation convention). This tangent vector operates on any function, ϕ, and essentially gives the rate of change of the function – or the *directional derivative* – in a direction which is defined by the numbers $(a^1, a^2, ..., a^n)$. We can define the addition of two tangent vectors in the obvious way

$$a^\mu \frac{\partial}{\partial x^\mu} + b^\mu \frac{\partial}{\partial x^\mu} = (a^\mu + b^\mu) \frac{\partial}{\partial x^\mu} \tag{91}$$

Mathematical Aside: With this straightforward definition of addition, a little formal mathematics easily shows that the set of all tangent vectors form what is called a *vector space*

Thus, the operator,

$$a^\mu \frac{\partial}{\partial x^\mu}$$

behaves like a vector, the components of which are $(a^1, a^2, ..., a^n)$. We therefore write

$$\vec{a} = a^\mu \frac{\partial}{\partial x^\mu} \tag{92}$$

The n operators $\frac{\partial}{\partial x^\mu}$ can be thought of as forming a set of basis vectors, $\{\vec{e_\mu}\}$, spanning the vector space of tangent vectors at P.

What exactly do these basis vectors represent? We can find a simple geometrical picture for the $\vec{e_\mu}$ by first crystallising the notion of a *curve*, C, defined on our manifold. Our intuitive notion of a curve is simply of a connected series of points on the manifold; in the mathematical literature, however, we call this a *path*, and the term *curve* is instead reserved for the particular case of a path which has been *parametrised*.

Thus, a curve is a function which maps an interval of the real line into the manifold. Putting this more simply, a curve is a path with a real number (s, say) associated with each point of the path; we call s the parameter of the curve. Note also that once we choose a coordinate system each point on the curve has coordinates, $\{x^\mu\}$, which may also be expressed as functions of the parameter, s, i.e.

$$x^\mu = x^\mu(s) \quad \mu = 1, ..., n \tag{93}$$

Once we specify our coordinate system, we can consider a particular set of curves which use the *coordinates themselves* as their parameter. For example, point P with coordinates $\{x^1, x^2, ..., x^n\}$ lies on the n curves which we obtain by allowing only the value of x^i to vary along the i^{th} curve ($i = 1, ..., n$) and fixing all other coordinate values to be equal to their values at P. (To visualise a simple example, think of circles of equal latitude and longitude on the 2-sphere manifold). The basis vector, $\vec{e_i} \equiv \frac{\partial}{\partial x^i}$ can be thought of simply as the *tangent* to the i^{th} curve. This geometrical picture is illustrated in Figure 8, again for the straightforward example of the 2-sphere. Note that the basis vectors $\vec{e_\phi}$ and $\vec{e_\theta}$ are *different* at points X and Y of the manifold.

And what of a more general curve in the manifold? Here we simply connect the notion, introduced above, of a tangent vector as a directional derivative to our straightforward geometrical picture of a tangent to a curve. Figure 9 shows a curve, with parameter s, and with tangent vectors drawn at points with different parameter values. Suppose the coordinates of the points on the curve are $\{x^\mu(s)\}$, for $\mu = 1, ..., n$. Then the components, T^μ, of the tangent vector with respect to the basis $\{\vec{e_\mu}\} \equiv \{\frac{\partial}{\partial x^\mu}\}$ are simply given by

$$T^\mu = \frac{dx^\mu}{ds} \tag{94}$$

To sum up, we can represent vectors as tangent vectors of curves in our manifold. Once we have specified our coordinate system, we can write down the components of a vector defined at any point of the manifold with respect to the natural basis generated by the derivative operators $\{\frac{\partial}{\partial x^\mu}\}$ at that point. A vector **field** can be defined by assigning a tangent vector at every point of the manifold, so that the components, $\{a^\mu\}$ now become functions of the coordinates, i.e.

$$\vec{a} = a^\mu(x^1, x^2, ..., x^n)\, \vec{e}_\mu \tag{95}$$

Figure 8

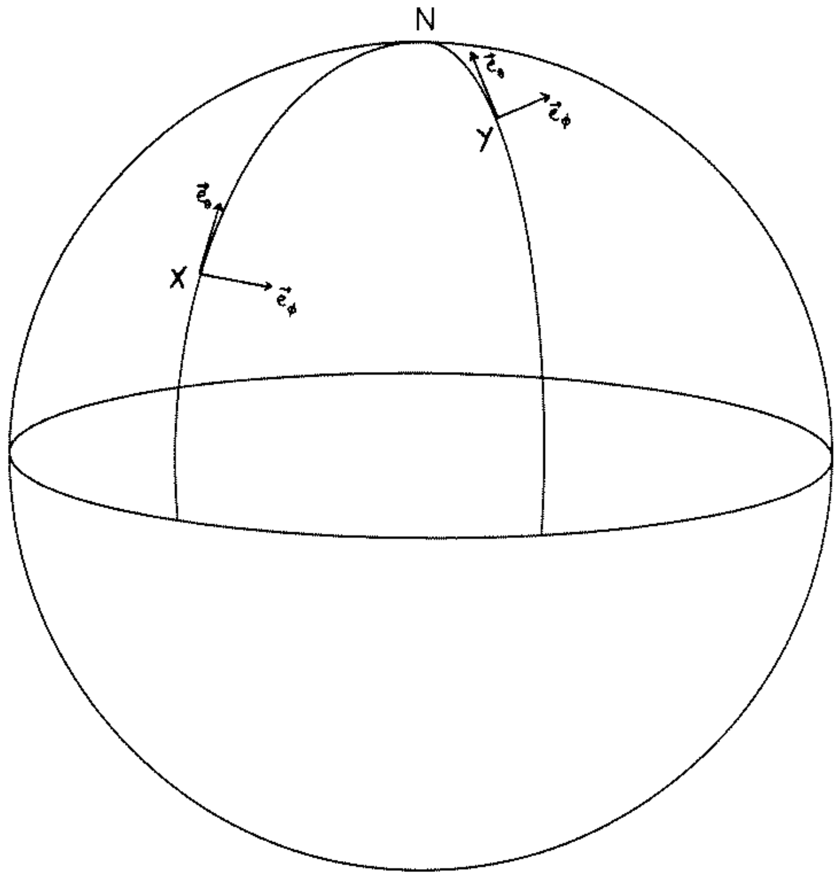
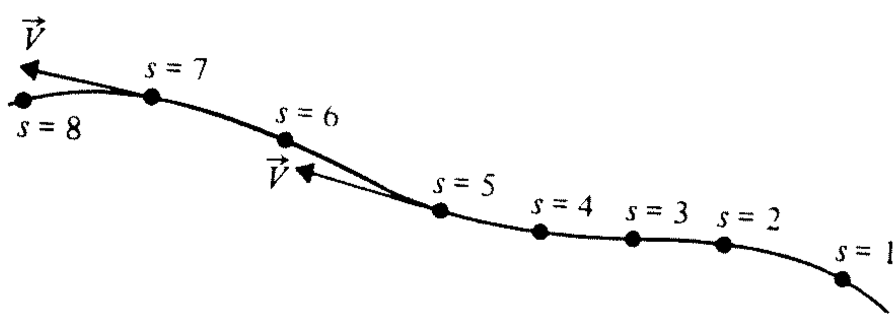


Figure 9



4.3.2 Transformation law for vectors

Suppose we change to a new coordinate system $\{x'^1, x'^2, ..., x'^n\}$. Our basis vectors are now

$$\vec{e'_\mu} \equiv \frac{\partial}{\partial x'^\mu} \tag{96}$$

How do the components, $\{a^1, a^2, ..., a^n\}$, transform in our new coordinate system? We have already derived this transformation law for displacements in equation (43). To see how the law arises within the framework of our tangent vector description, let the vector $\vec{a}$ operate on an arbitrary scalar function, ϕ. Then

$$\vec{a}(\phi) = a^\nu \frac{\partial \phi}{\partial x^\nu} \tag{97}$$

By the chain rule for differentiation we may write this as

$$\vec{a}(\phi) = a^\nu \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial \phi}{\partial x'^\mu} \tag{98}$$

However, if we write $\vec{a}$ directly in terms of coordinate basis $\{\vec{e'_\mu}\} = \{\frac{\partial}{\partial x'^\mu}\}$, we have

$$\vec{a}(\phi) = a'^\mu \frac{\partial \phi}{\partial x'^\mu} \tag{99}$$

Comparing equation (98) with (99) it is evident that

$$a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} a^\nu \tag{100}$$

Thus the components of the tangent vector transform according to equation (100). We call this equation the transformation law for a **contravariant vector**, and say that the components of $\vec{a}$ transform **contravariantly**. (The term 'contravariant' is used to distinguish these vectors from another type of geometrical object – covariant vectors or 'covectors' – which we will meet in the next subsection. The more modern name for covariant vectors, however, is 'one-forms', and we will generally adopt that name in order to avoid this source of ambiguity). We denote the components of a contravariant vector as superscripts.

As we remarked above, equation (100) is the same transformation law as we introduced in equation (43) to describe the transformation of a small displacement, $\Delta\vec{x}$. To fix these ideas within our tangent vector framework, consider two neighbouring points P and Q of the manifold with coordinates $\{x^\mu\}$ and $\{x^\mu + dx^\mu\}$ respectively. Here the dx^μ are considered to be infinitesimal. In another, primed, coordinate system P and Q have coordinates $\{x'^\mu\}$ and $\{x'^\mu + dx'^\mu\}$. Since

$$x'^\mu = x'^\mu(x^1, x^2, ..., x^n) \tag{101}$$

it follows that

$$\begin{aligned} x'^\mu + dx'^\mu &= x'^\mu(x^1 + dx^1, x^2 + dx^2, ..., x^n + dx^n) \\ &= x'^\mu(x^1, x^2, ..., x^n) + \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \end{aligned} \tag{102}$$

It then follows that

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \tag{103}$$

Thus, the infinitesimals $\{dx^\mu\}$ transform as contravariant components. One can refer to $\{dx^1, dx^2, ..., dx^n\}$ as being a contravariant vector.

Equations (100) and (103) are two examples of the prototype transformation law for *any* contravariant vector. Any quantity, A^μ, with n components which can be evaluated in any coordinate system, and which transform in the same way as dx^μ, according to the transformation law of equation (103), is called a contravariant vector. (Strictly, in the coordinate-free approach one would talk of $\vec{A} = A^\mu \vec{e}_\mu$ being the vector, and A^μ its components). Thus for any contravariant vector

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \tag{104}$$

4.3.3 One-Forms

What is the relationship between the basis vectors $\vec{e'_\mu}$ and $\vec{e_\mu}$ in the primed and unprimed coordinate systems? From equation (96) we have

$$\vec{e'_\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \vec{e_\nu} \tag{105}$$

(which is equivalent to equation (75) of Section 3 above).

Thus we see that the basis vectors do *not* transform in the same way as the components of a contravariant vector. This should not be too surprising, since the transformation of a basis and the transformation of components are different things: the former is the expression of *new* vectors in terms of *old* vectors; the latter is the expression of the *same* vector in terms of a new basis.

In fact, the form of the transformation in equation (105) is the same as the transformation law for another type of geometrical object, which we call a **covariant vector**, **covector**, or (in more modern literature) a **one-form**. Any quantity, A_μ, with n components which can be evaluated in any coordinate system, is said to be a one-form if the components transform according to the equation

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \tag{106}$$

(The old-fashioned name for a one-form, a covariant vector, arose because the components transform in the *same* way as (i.e. 'co') basis vectors, while the components of a contravariant vector transform in the *opposite* way (i.e. 'contra') to basis vectors).

One can simply regard equation (106) as *defining* a one-form. Many modern textbooks on differential geometry, however, begin by defining a one-form as a linear mapping which acts on a vector to give a real number. (Starting from this definition one can then arrive at equation (106).

One-forms are usually denoted by a tilde above a symbol, just as vectors are denoted by an arrow above a symbol. Thus $\tilde{p}(\vec{a})$ is a real number. If $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$ are one-forms, then we define their addition and scalar multiplication properties via the relations

$$\tilde{s} = \tilde{p} + \tilde{q} \equiv \tilde{s}(\vec{a}) = \tilde{p}(\vec{a}) + \tilde{q}(\vec{a}) \tag{107}$$

$$\tilde{r} = \alpha\tilde{p} \equiv \tilde{r}(\vec{a}) = \alpha\tilde{p}(\vec{a}) \tag{108}$$

for any vector, $\vec{a}$. With these rules the set of all one-forms is a *vector space*, which we call the *dual* space of of the vector space of contravariant vectors. There is a close relationship between these two vector spaces. For example, there is a natural basis of one-forms which we denote $\tilde{\omega}^\alpha \equiv \tilde{dx}^\alpha$, which are related to the basis *vectors*, $\vec{e}_\beta$, by the equation

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \tag{109}$$

With this basis we can write any one-form in terms of components

$$\tilde{p} = p_\alpha \tilde{\omega}^\alpha \tag{110}$$

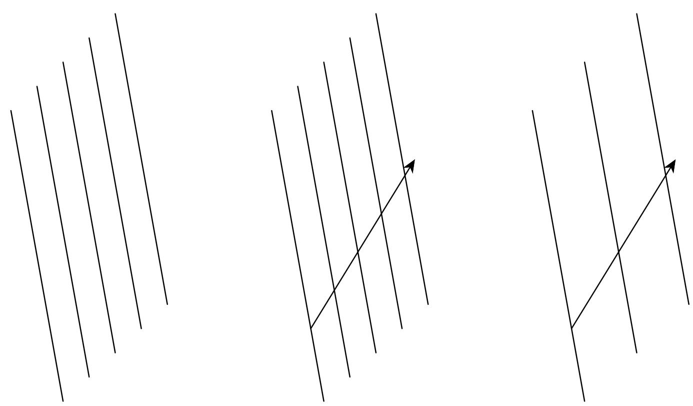
where $p_\alpha = \tilde{p}(\vec{e}_\alpha)$.

4.3.4 Picture of a One-Form

If we need a picture to represent a vector we usually think of an arrow. It is helpful to have a picture of a one-form as well. First of all, it is not an arrow. Its picture must reflect the fact that it maps vectors into real numbers. The picture generally used by mathematicians is shown in Figure (10). Here the one-form consists of a series of surfaces in the neighbourhood of a point in the manifold [panel (a)]. The 'magnitude' of the one form is given by the spacing between the surfaces: the smaller the spacing, the larger the magnitude of the one-form. Thus the one-form in panel (b) has a larger magnitude than the one-form in panel (c). In this picture, the real number produced when a one-form acts on a vector is the number of surfaces that the vector crosses; hence, the closer the spacing the larger the number. The one-form doesn't define a unique direction, since it is not a vector. Rather it defines a way of 'slicing' the manifold.

In order to justify this picture we shall look at a particular one-form: the *gradient*.

Figure 10



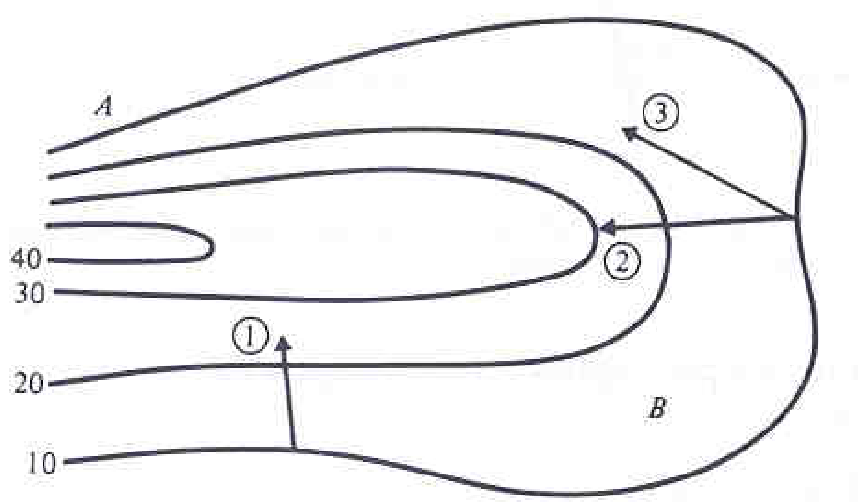
As in Section 4.3.1, consider a scalar field, ϕ, defined everywhere on a Riemannian manifold. At point P, with coordinates $\{x^1, x^2, ..., x^n\}$, form the derivatives $\{\frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}, ..., \frac{\partial\phi}{\partial x^n}\}$. Suppose we now change to a new, primed, coordinate system, in which the point P has coordinates $\{x'^1, x'^2, ..., x'^n\}$. From the chain rule for differentiation we have, for each μ,

$$\frac{\partial\phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi}{\partial x^\nu} \tag{111}$$

Comparing equation (111) with equation (106) we see that the gradient of ϕ fits our definition of a one-form, since its components transform according to the correct transformation law. It is usually denoted by $\tilde{d}\phi$. (In elementary courses on calculus and geometry the gradient is usually introduced as a *vector*, i.e. with a defined direction. We will see in the next section why this is justifiable, at least for Euclidean space).

The gradient enables us to justify our picture of a one-form, through the following simple example. Figure 11 is part of a topographical map, showing contours of equal elevation. If h is the elevation, then the gradient, $\tilde{d}h$, is clearly largest in an area like A, where the contour lines are closest together, and smallest near B, where the contour lines are spaced far apart. Moreover, suppose one wanted to know how much elevation a walk between two points would involve. One would lay out on the map a vector between the starting and finishing points, and the number of contours which the line crossed would give the change in elevation. For example, vector (1) crosses 1.5 contours, while vector (2) crosses 2 contours. Vector (3) starts from the same point as vector (2), but goes in a different direction and winds up only about 0.75 contours higher.

Figure 11



4.4 Tensors

Having defined what me mean by vectors and one-forms, in terms of how their components transform under a general coordinate transformation, we can now extend our definition to the more general class of geometrical object which we call *tensors*.

A tensor of type (l, m), defined on an n dimensional manifold, is a linear operator which maps l one-forms and m (contravariant) vectors into a real number (i.e. scalar). Such a tensor has a total of n^{l+m} components.

The transformation law for a general (l, m) tensor follows from its linearity, and from the transformation laws for a vector and one-form, in order that the scalar quantity obtained when the tensor operates on l one-forms and m vectors is independent of one's choice of coordinate system. We can write this general transformation law as follows

$$A'^{u_1\, u_2\, \ldots\, u_l}_{r_1\, r_2\, \ldots\, r_m} = \frac{\partial x'^{u_1}}{\partial x^{t_1}} \cdots \frac{\partial x'^{u_l}}{\partial x^{t_l}} \frac{\partial x^{q_1}}{\partial x'^{r_1}} \cdots \frac{\partial x^{q_m}}{\partial x'^{r_m}} A^{t_1\, t_2\, \ldots\, t_l}_{q_1\, q_2\, \ldots\, q_m} \tag{112}$$

This somewhat intimidating equation appears much more straightforward for some specific cases. First note that a contravariant vector is in fact a $(1,0)$ tensor (since it operates on a one-form to give a scalar). To see this, substitute $l = 1$, $m = 0$ into equation (112) and recover equation (104). Similarly a one-form is a $(0,1)$ tensor (and more trivially a *scalar* is a $(0,0)$ tensor).

A $(2,0)$ tensor, say T^{ij}, is called a *contravariant* tensor of rank 2 and transforms according to the transformation law

$$T'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} T^{kl} \tag{113}$$

A $(0,2)$ tensor, say B_{ij}, is called a *covariant* tensor of rank 2, and transforms according to the law

$$B'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} B_{kl} \tag{114}$$

An important example of a $(0,2)$ tensor is the *metric tensor*, $g_{\alpha\beta}$, which we will discuss in more detail in the next section. We already met the metric tensor, $\eta_{\alpha\beta}$, in Section 3, in its simplified form for the Minkowski metric of special relativity. Recall from equations (47) and (48) that we gave the transformation law for Lorentz matrices which ensured the invariance of the interval, Δs^2, in different Lorentz frames. We can now see that the form of equations (47) and (48) is consistent with $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$ being a second rank covariant and contravariant tensor respectively. Note also that our expression, in equation (57), for the magnitude (i.e. scalar product) of the four velocity now makes more sense: we expect a second rank covariant tensor ($\eta_{\alpha\beta}$) operating on two contravariant vectors (v^α, v^β) to give a real number.

A tensor which has both upper and lower indices, which means that it has both contravariant and covariant terms in its transformation law, is known as a *mixed tensor*. The simplest example (after the trivial case of a $(0,0)$ tensor) is a $(1,1)$ tensor, D^i_j, say. Its transformation law is

$$D'^i_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} D'^k_l \tag{115}$$

An important example of a $(1, 1)$ tensor is the Kronecker delta, δ^i_j, which we met first in Section 2.

Exercise: Show that δ^i_j transforms as a $(1, 1)$ tensor.

One way to construct a $(1, 1)$ tensor is to take the *product* (formally the *outer product*) of a vector and one-form, i.e.

$$D^i_j = A^i B_j \tag{116}$$

(Clearly we may generalise this procedure to mixed tensors of higher rank).

4.4.1 Contraction of tensors

We can also take another kind of product in equation (116), known as the **inner product**, or **contraction** of a vector and one-form; i.e. we form the quantity $A^i B_i$ (where, as usual, the summation convention is implied). This quantity can easily be shown to be an invariant or scalar in the sense that

$$A'^j B'_j = A^i B_i \tag{117}$$

Exercise: Verify equation (117)

We can generalise the operation of contraction to the case of any two tensors, and over an arbitrary number of indices, provided that an equal number of upper and lower indices are selected. In general, contraction over k indices will produce from a tensor of type (l, m) a new tensor of type $(l - k, m - k)$. For example, the contraction of the two tensors G^{ijk}_{lm} and R^s_{tu} over the indices i and t, j and u and l and s will give the $(1, 1)$ tensor $G^{ijk}_{lm} R^l_{ij}$, where now only the indices k and m are free indices.