3 Special relativity

To understand GR we need to spend a little time on SR. This will also give us the opportunity to discuss covariance in SR, and introduce some notation that will be useful in GR. Bear in mind that SR is essentially the same as GR when the spacetime is globally flat. The notion of a **spacetime manifold** arises quite naturally in special relativity. **Events** are the points of this manifold, and 4 coordinates are required to specify any particular event. These are usually taken to be a time coordinate, and three spatial coordinates. We shall refer to these coordinates as x^0, x^1, x^2, x^3, where x^0 denotes the time coordinate. We shall also measure time in the same units as space. Usually we shall take the spatial unit to be metres, in which case time is also measured in metres – i.e. the time in which light travels one metre. (A metre will thus be $(3 \times 10^8)^{-1}$ seconds). Evidently in these units, the speed of light is 1, and we write $c = 1$. (It is also common to measure the time coordinate in terms of the variable ct).

3.1 Invariant distance in spacetime

Between any two events an invariant "distance" can be defined. Thus in an inertial frame using cartesian coordinates, the "distance" or interval between event $A(x^0_A, x^1_A, x^2_A, x^3_A)$ and event $B(x^0_B, x^1_B, x^2_B, x^3_B)$ is given by

$$s^2 = -(x^0_A - x^0_B)^2 + (x^1_A - x^1_B)^2 + (x^2_A - x^2_B)^2 + (x^3_A - x^3_B)^2 \tag{30}$$

Another inertial observer using coordinates $\{x'^0, x'^1, x'^2, x'^3\}$ would obtain the same value when evaluating

$$s'^2 = -(x'^0_A - x'^0_B)^2 + (x'^1_A - x'^1_B)^2 + (x'^2_A - x'^2_B)^2 + (x'^3_A - x'^3_B)^2 \tag{31}$$

For this reason the quantity s^2 in equation (30) is called an **invariant**.

Rather than two finitely separated events, consider two infinitesimally separated events whose coordinates in inertial frame S are (x^0, x^1, x^2, x^3) and $(x^0 + \Delta x^0, x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 + \Delta x^3)$ respectively. The interval between the two events is given by

$$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 \tag{32}$$

3.2 The metric in special relativity

We can write equation (32) in the form

$$\Delta s^2 = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta \tag{33}$$

where $\eta_{00} = -1$, $\eta_{11} = 1$, $\eta_{22} = 1$, $\eta_{33} = 1$ and all other components are zero. $\eta_{\alpha\beta}$ will appear later in GR as the metric tensor, $g_{\alpha\beta}$. Sometimes we write the metric tensor in matrix form, i.e.

$$\mathbf{N} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \tag{34}$$

In equation (33), $\eta_{\alpha\beta}$ is diagonal and constant because:-

- we only consider spatially cartesian coordinates, and inertial frames and
- in SR spacetime is **flat**

We define $\eta^{\alpha\beta}$ to be the inverse of $\eta_{\alpha\beta}$, i.e.

$$\eta^{\alpha\lambda}\eta_{\lambda\beta} = \delta^\alpha_\beta \tag{35}$$

where δ^α_β is the Kronecker delta defined in equation (24).

3.3 Spacelike, timelike and null intervals

Δs has the dimensions of metres.

$$\begin{aligned} \Delta s^2 &> 0 \text{ the interval is spacelike} \\ \Delta s^2 &< 0 \text{ the interval is timelike} \\ \Delta s^2 &= 0 \text{ the interval is null} \end{aligned} \tag{36}$$

- When the interval is **spacelike** a Lorentz frame can be found in which the two events are simultaneous
- When the interval is **timelike** a Lorentz frame can be found in which the two events have the same spatial coordinates
- When the interval is **null** each event lies on the light cone of the other.

3.4 Proper time

If an interval is timelike, $\Delta\tau$ denotes the proper time between the two events and is given by $\Delta\tau^2 = -\Delta s^2$. The set of all events in the past and future of a particle is its **worldline**. From equation (32) it follows that

$$\Delta\tau = \{(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2\}^{1/2} \tag{37}$$

Clearly, since

$$\Delta x^1/\Delta x^0 = v^1$$

etc., we may write

$$\Delta\tau = \left(1 - (v)^2\right)^{1/2} \Delta x^0 = \gamma(v)^{-1}\Delta x^0 \tag{38}$$

where $(v)^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$.

Thus between any two events on its worldline the particle will record a time of

$$\tau = \int_{E_1}^{E_2} d\tau \tag{39}$$

If we evaluate this time in any inertial frame this may be written

$$\tau = \int_{E_1}^{E_2} \left(1 - (v)^2\right)^{1/2} dt = \int_{E_1}^{E_2} \gamma(v)^{-1} dt \tag{40}$$

3.5 The Lorentz group

Consider inertial frames S, with coordinate system $\{x^0,\ x^1,\ x^2,\ x^3\}$, and S', with coordinate system $\{x'^0, x'^1, x'^2, x'^3\}$. ($S'$ travels at uniform velocity w.r.t. S, but their axes are not necessarily aligned.) We may write in matrix notation

$$\mathbf{x}' = \mathbf{\Lambda x} + \mathbf{c} \tag{41}$$

where $\mathbf{c}$ is a constant and $\mathbf{\Lambda}$ is a 4×4 Lorentz matrix. Noting that Λ^μ_ν is the element in the μ row and ν column of the matrix $\mathbf{\Lambda}$, in index notation equation (41) becomes

$$x'^\mu = \Lambda^\mu_\alpha x^\alpha + c^\mu \tag{42}$$

If the origins are chosen to coincide when $x^0 = x'^0 = 0$ then $c^\mu = 0$. We may write

$$\Delta x'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha}\Delta x^\alpha \tag{43}$$

From equation (42) it is now obvious that

$$\Lambda^{\mu}_{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \tag{44}$$

We can therefore write equation (43) in the more suggestive form

$$\Delta x'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} \tag{45}$$

To ensure the invariance of Δs^2, $\mathbf{\Lambda}$ has to satisfy

$$\mathbf{\Lambda}^{\mathrm{T}} \mathbf{N}' \mathbf{\Lambda} = \mathbf{N} \tag{46}$$

Matrices which satisfy equation (46) are called **Lorentz matrices**. In component form this condition may be written

$$\eta'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \eta_{\alpha\beta} \tag{47}$$

It also follows that

$$\eta'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \eta^{\alpha\beta} \tag{48}$$

Exercise: Verify equations (47) and (48).

If S' moves along the x^1 axis of S, the Lorentz matrix may simply be written

$$\begin{pmatrix} \gamma(v) & -v\gamma(v) & 0 & 0 \\ -v\gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

The set of transformations given by equation (41) forms a group. Similarly the set of Lorentz matrices forms a group. The latter may easily be demonstrated by showing that if $\mathbf{\Lambda_1}$ and $\mathbf{\Lambda_2}$ satisfy equation (46) then so does $\mathbf{\Lambda_1 \Lambda_2}$. The inverse of $\mathbf{\Lambda}$ also satisfies equation (46) as does the unit matrix, $\mathbf{I}$.

Exercise: Show that the Lorentz matrices form a group.

Exercise: Show that the transformations in equation (41) form a group, of which the Lorentz group is a subgroup. (This larger group is called the Poincaré group.)

3.6 Four vectors

The four velocity of a particle is defined to be

$$v^\mu = \frac{dx^\mu}{d\tau} \tag{50}$$

Strictly speaking equation (50) defines only the components in some coordinate system of the four velocity, $\vec{v}$. $\vec{v}$ exists independently of all coordinate systems. There is a natural set of basis vectors associated with any given coordinate system (we shall see this later), and for the usual cartesian coordinates $\{x^0,\ x^1,\ x^2,\ x^3\}$ in inertial frame, S, these basis vectors, $\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ correspond to unit displacements along the x^0, x^1, x^2 and x^3 axes. Thus we may write

$$\vec{v} = v^\mu \vec{e}_\mu \tag{51}$$

The components of the four velocity must transform under a Lorentz transformation in the same way as Δx^μ, since $\Delta\tau$ is an invariant. Writing out the components in a given Lorentz frame, we have

$$v^0 = \frac{dx^0}{d\tau} = \left(1 - (v)^2\right)^{-1/2} = \gamma(v) \tag{52}$$

$$v^1 = \frac{dx^1}{d\tau} = \frac{dx^1}{dx^0}\frac{dx^0}{d\tau} = \gamma(v)v^x \tag{53}$$

$$v^2 = \frac{dx^2}{d\tau} = \frac{dx^2}{dx^0}\frac{dx^0}{d\tau} = \gamma(v)v^y \tag{54}$$

$$v^3 = \frac{dx^3}{d\tau} = \frac{dx^3}{dx^0}\frac{dx^0}{d\tau} = \gamma(v)v^z \tag{55}$$

where v^x, v^y and v^z are the components of the three velocity, $\mathbf{v}$. Sometimes one sees written $v^\mu = \gamma(1, \mathbf{v})$, although this is a bit of a 'mixed metaphor'.

Notice that

$$v'^\mu = \frac{dx'^\mu}{d\tau} = \Lambda^\mu_\nu v^\nu = \frac{\partial x'^\mu}{\partial x^\nu} v^\nu \tag{56}$$

Exercise: In frame S a particle has four velocity $\gamma(u)(1, u, 0, 0)$. If frame S' moves at velocity, v, along x-axis of S, what are the components of the four velocity in frame S'? Comment on your answer.

The magnitude of the four velocity, $(\vec{v}.\vec{v})^{1/2}$ is defined via

$$\vec{v}.\vec{v} = \eta_{\alpha\beta} v^\alpha v^\beta \tag{57}$$

$\vec{v}.\vec{v}$ is often written v^2, which can be confusing. Notice that $\vec{v}.\vec{v}$ as given by equation (57) is an invariant – i.e. one would obtain exactly the same value when evaluated in any inertial frame. It is easy to verify that

$$\vec{v}.\vec{v} = \eta_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \left(-1 + (v)^2\right) = -1 \tag{58}$$

Exercise: Show that

$$\eta_{\alpha\beta} v^\alpha v^\beta = \eta'_{\alpha\beta} v'^\alpha v'^\beta$$

3.6.1 Covariant Components

As a precursor to our discussion of *one-forms* in Section 4, we can introduce what are called the *covariant* components, v_μ, defined by

$$v_\mu = \eta_{\mu\alpha} v^\alpha \tag{59}$$

Thus $v_0 = -v^0$, $v_i = v^i$, where i takes the values from 1 to 3. Hence we may write

$$\vec{v}.\vec{v} = \eta_{\alpha\beta} v^\alpha v^\beta = v_\mu v^\mu \tag{60}$$

3.6.2 The four momentum

So far we have dealt with the four velocity. We can simply define the four momentum of a particle as

$$\vec{p} = m\vec{v} \tag{61}$$

where m is the rest mass of the particle. In component notation equation (61) becomes

$$p^\mu = m v^\mu \tag{62}$$

and is evidently valid in all inertial frames.

3.6.3 The four acceleration

We define the four acceleration as

$$\vec{a} = \frac{d\vec{v}}{d\tau} = \frac{d}{d\tau} v^\mu \vec{e_\mu} \tag{63}$$

In an inertial and cartesian frame in which

$$\frac{d\vec{e}_\mu}{d\tau} = 0$$

we may write

$$\vec{a} = \frac{dv^\mu}{d\tau}\vec{e_\mu} \tag{64}$$

In this Lorentz frame we may write $\vec{a}$ in component form as

$$a^\mu = \frac{dv^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} \tag{65}$$

The (components of the) four acceleration may also be defined as

$$a^\mu = \frac{dv^\mu}{d\tau} \tag{66}$$

and must transform in the same way as v^μ.

p^μ and a^μ transform in exactly the same way as v^μ. $\vec{p}$ and $\vec{a}$ provide examples of four vectors. In the same way as before, we can define the *covariant* components of these four vectors as

$$p_\mu = \eta_{\mu\alpha}p^\alpha \tag{67}$$

$$a_\mu = \eta_{\mu\alpha}a^\alpha \tag{68}$$

Exercise: Show that $p_\mu p^\mu = -m^2$.

The relativistic version of Newton's laws now has to be of the form

$$\vec{f} = m\frac{d\vec{v}}{d\tau} \tag{69}$$

where $\vec{f}$ is a four force; i.e. it has to be a four vector. In a Lorentz frame, equation (69) in component form becomes

$$f^\mu = m\frac{d^2x^\mu}{d\tau^2} \tag{70}$$

Conservation of four momentum will read

$$\sum_{i=1}^{N} \vec{p_{(i)}} = \vec{P}_{\text{tot}} \tag{71}$$

where $\vec{P}_{\text{tot}}$ is a constant four vector, and the sum is over N particles. In component form equation (71) becomes

$$\sum_{i=1}^{N} p^\mu_{(i)} = P^\mu_{\text{tot}} \quad \text{for } \mu = 0,1,2,3 \tag{72}$$

This evidently could also be written in terms of covariant components. (Note that in equation (72) the subscript '(i)' is simply a label to denote the i^{th} particle and should not be confused with the covariant components, p_μ). Equations (69) and (71) are explicitly independent of all coordinate systems. Equations (72), on the other hand, which are in component form, will be valid in all frames, although the components p_a^μ will be different according to in which frame they are evaluated.

Equation (70), with a^μ as $d^2x^\mu/d\tau^2$ is valid in all inertial frames, but not true in arbitrary coordinate systems. Equation (70) is covariant under Poincare (and Lorentz) transformations, but **not** under more general transformations.

3.7 Generalising to tensors

So far we have only discussed how four vectors transform, but we can easily extend our discussion to more general geometrical objects known as **tensors**. We shall usually just talk about the components of a tensor, eg $F_{\mu\nu}$, and their transformation properties, and sometimes simply refer to these components as the tensor itself. Mathematically this is not entirely correct, but is expedient and should not lead to confusion in the context of this course.

Thus we can simply *define* a tensor in terms of its transformation properties. (We will discuss this in more detail in Section 4).

A simple example of a tensor would be $C^{\mu\nu} = a^\mu b^\nu$. Under a Lorentz transformation this tensor would transform as

$$C'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}C^{\alpha\beta} \tag{73}$$

An equation of the form

$$C^{\mu\nu} = D^{\mu\nu} \tag{74}$$

where both $\mathbf{C}$ and $\mathbf{D}$ are tensors, will necessarily be valid in all Lorentz frames. Such equations are called tensor equations, or covariant equations.

The basis vectors $\vec{e}_\mu$ corresponding to coordinates $\{x^0, x^1, x^2, x^3\}$ are related to basis vectors $\vec{e'}_\mu$ corresponding to coordinates $\{x'^0, x'^1, x'^2, x'^3\}$, via the transformation law

$$\vec{e}_\mu = \frac{\partial x'^\mu}{\partial x^\mu}\vec{e'}_\mu \tag{75}$$

This makes sense, as we want four vectors, $\vec{a}$, etc to be independent of coordinates, and evidently

$$\vec{a} \;=\; a^\mu \vec{e}_\mu = \; a^\mu \frac{\partial x'^\mu}{\partial x^\mu} \vec{e'}_\mu = a'^\mu \vec{e'}_\mu \tag{76}$$

There is a close relationship between the basis vector $\vec{e}_\mu$ and the operator $\partial/\partial x^\mu$. We shall see this when we consider curved spaces. Note that it has the same 'transformation' properties as $\vec{e}_\mu$, i.e.

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\mu}{\partial x^\mu} \frac{\partial}{\partial x'^\mu} \tag{77}$$