2 Covariance

In SR all inertial frames are equally valid; accordingly physical laws should be expressible in a manner such that they are **covariant** under what is termed the *Lorentz group* – i.e. the group of all Lorentz transformations [1].This simply means that the same physical laws apply in all Lorentz frames. In GR we go one step further: physical laws should remain valid under *all* coordinate transformations – we call this the **principle of general covariance**.

2.1 Covariance in Newton's laws

Consider Newton's law of motion to illustrate the notion of covariance.

$$\vec{F} = m\vec{a} \tag{11}$$

is a vector equation. Vectors $\vec{F}$ and $\vec{a}$ exist independently of our choice of coordinate system, and so the physical law is independent of the coordinate system. It is a *coordinate free* description.

Alternatively equation (11) can be written in component form. Choosing basis vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ we may write

$$\vec{F} = \sum_{i=1}^{3} F^i \vec{e}_i \tag{12}$$

and

$$\vec{a} = \sum_{i=1}^{3} a^i \vec{e}_i \tag{13}$$

Thus equating components, equations (12) and (13) give

$$F^i = ma^i \tag{14}$$

Of course the components of $\vec{F}$ and $\vec{a}$ will have different values if different basis vectors are chosen. (We shall see that a change in coordinate system will also generate a change in basis vectors). So the *components* of the vector equation are

[1] We use the term *group* here in its precise mathematical sense: a group is a set and a binary operation which acts on the elements of that set such that certain *group axioms* are satisfied. No formal knowledge of group theory is required for this course, however, and the further discussion of the group properties of Lorentz transformations in Section 3 is non-examinable.

not invariant, and will change depending on one's coordinate system, even though the physical law **is** invariant, and holds regardless of one's choice of coordinate system. We say that the vector equation in component form provides a *covariant* description of Newton's law of motion; the prefix 'co' reminds us that the numerical *value* of the *co*mponents depends on our choice of *co*ordinate system.

2.2 Summation convention

Instead of constantly writing

$$\sum_i F^i \vec{e}_i \tag{15}$$

we shall simply write

$$F^i \vec{e}_i \tag{16}$$

Thus, where one encounters repeated indices (upper-lower or lower-upper), this implies summation. (Einstein once commented that this discovery was one of his greatest achievements!).

Example 1:

$$\sum_i a^i\, b_i \text{ is written } a^i\, b_i \quad \text{or} \quad a^j\, b_j \tag{17}$$

Example 2

$$\sum_j \sum_i A^{ijk}\, B_{ijl} \text{ is written } A^{ijk}\, B_{ijl} \tag{18}$$

(N.B. "i" and "j" are called dummy indices; the choice of letter is unimportant)

2.3 Change of basis

We can also use Newton's law to illustrate how coordinate bases transform. Let us express the vector $\vec{F}$ in a new basis $\{\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\}$, i.e.

$$\vec{F} = F'^k \vec{e'_k} \tag{19}$$

Of course we can also write the basis vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in terms of this new basis, i.e.

$$\vec{e}_j = A^k_j \vec{e'_k} \tag{20}$$

where $A^k_j \equiv \mathbf{A}$ is a non-singular matrix, since $\{\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\}$ is also a coordinate basis.

Conversely we may write $\{\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\}$ in terms of basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, i.e.

$$\vec{e'_k} = B^l_k \vec{e}_l \tag{21}$$

Substituting equation (21) into equation (20) it follows that

$$\vec{e}_j = A^k_j B^l_k \vec{e}_l \tag{22}$$

or

$$A^k_j B^l_k = \delta^l_j \tag{23}$$

where

$$\delta^l_j = 1 \text{ when } j = l \text{ and } 0 \text{ when } j \neq l \tag{24}$$

δ^l_j is called the kronecker delta. Condition (23) we may also write in matrix form as

$$\mathbf{A}\,\mathbf{B} = \mathbf{I} \;\; \mathit{or} \;\; \mathbf{B} = \mathbf{A}^{-1}\,, \quad \mathbf{A} = \mathbf{B}^{-1} \tag{25}$$

Clearly

$$\vec{F} = F^i \vec{e}_i = F^i A^k_i \vec{e'_k} \tag{26}$$

However we also have $\vec{F} = F'^k \vec{e'_k}$, so necessarily

$$F'^k = A^k_i F^i \tag{27}$$

This defines the transformation law for the components of a vector. Evidently the components of $\vec{a}$ transform in the same way. Thus Newton's law in component form becomes

$$F^i = ma^i \tag{28}$$

In basis $\{\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\}$ Newton's law takes precisely the same form, i.e.

$$F'^i = ma'^i \tag{29}$$

and for that reason equation (28) is called covariant, since it takes the same **form** in all coordinate systems, even though the components will, in general, be different in different coordinate systems.