1 Introduction: Foundations of GR

General relativity (GR) explains gravitation as a consequence of the curvature of spacetime. In turn spacetime curvature is a consequence of the presence of matter. Spacetime curvature affects the movement of matter, which reciprocally determines the geometric properties and evolution of spacetime. We can sum this up neatly as follows:-

> *"Spacetime tells matter how to move,*
> *and matter tells spacetime how to curve"*

By the completion of this course, we will see that the "geometry" or curvature properties of spacetime can be described by what is called the **Einstein tensor**, $G_{\mu\nu}$, and the matter/energy properties by the **energy-momentum tensor**, $T_{\mu\nu}$. Einstein's theory of general relativity then states that these two tensors are proportional to each other, i.e.:-

$$G_{\mu\nu} = kT_{\mu\nu} \tag{1}$$

(where k is a constant, the numerical value of which depends on Newton's gravitational constant.)

Equation (1) is in fact a set of 16 equations, known as Einstein's field equations of general relativity, and basically contains everything we need to know in this course! These ten lectures are all about understanding where equation (1) comes from, how it is derived, what the symbols represent and what it means physically. To do that we first need to develop specific mathematical tools (known as tensor analysis) which can describe curved spacetime. We will then derive Einstein's equations, and show that they reduce to familiar Newtonian gravity in the non-relativistic limit. Specific astronomical applications to planetary orbits, stellar structure, neutron stars and black holes and gravitational waves will be covered in Gravitation and Relativity II.

GR is, as the name implies, a generalisation of **special relativity** (SR). In SR Einstein attempted to formulate the known laws of physics so that they would be valid in all **inertial frames**. If one accepts that the speed of light is a constant in all such frames, then one has also to accept there is no absolute time. Similarly, the distance between points in 3-D space is no longer invariant, but will be measured differently by different inertial observers. Newtonian gravitation is thus inherently non-relativistic since it describes the gravitational force between two masses as acting instantaneously, and as depending on the distance separating the two masses. Different inertial observers would not agree about either point, and so would not agree about the force of gravity between the two masses. In developing general relativity, Einstein set out to describe gravity in a manner that could be defined consistently by *any* observer, no matter how they were moving relative to any other observer. Einstein realised, however, that this seemed to imply something very profound: that gravity and acceleration are fundamentally equivalent – an idea which he enshrined in the **principles of equivalence**.

1.1 Equivalence principles

The linch-pins of GR are (i) the equivalence principles and (ii) the principle of general covariance. Let us first consider the former.

1.1.1 The weak principle of equivalence

This simply states that the inertial mass, m_I, and the gravitational mass, m_G, of a body are equal. Thus in a gravitational potential, ϕ, the force acting on a body is

$$\vec{F} = -m_G \, \nabla\phi \tag{2}$$

On the other hand according to Newton's laws of motion,

$$\vec{F} = m_I \, \vec{a} \tag{3}$$

It follows that in a gravitational field, all test bodies will accelerate at the same rate, $\vec{a} = -\nabla\phi$.

GR incorporates this result by demanding that *test particles* have worldlines that are *geodesics* in curved spacetime. Hence the worldline is independent of the mass of the test particle and depends only on the geometry of spacetime. Translating back into Newtonian language, this means that all bodies accelerate in a gravitational field at the same rate, regardless of their mass.

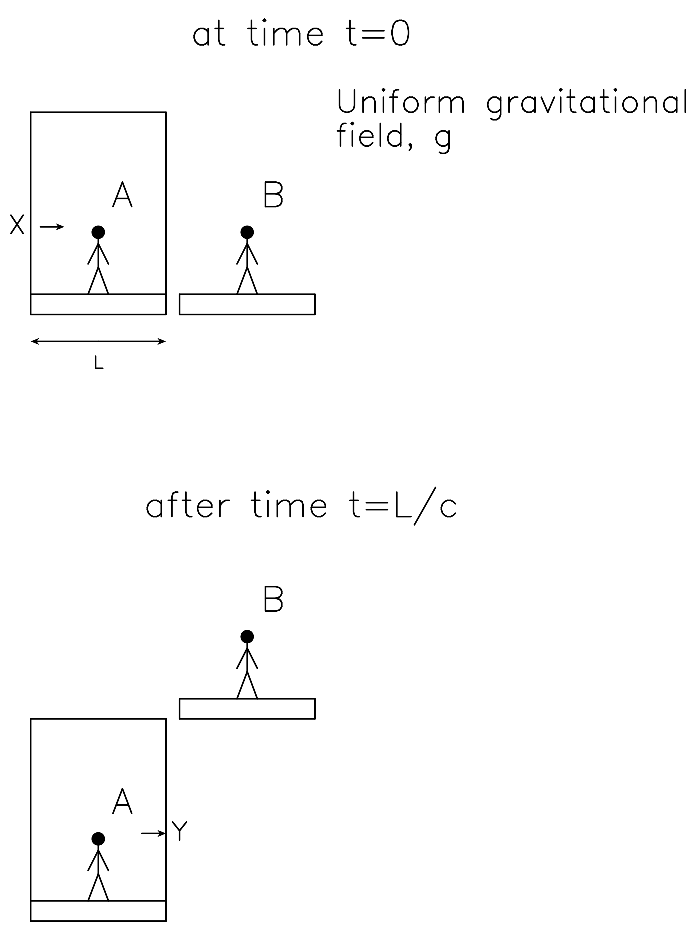
1.1.2 The strong principle of equivalence

The strong principle goes further and states that locally, i.e. in a local inertial frame (or free-falling frame), all physical phenomena are in agreement with special relativity. There are two important and immediate consequences of this principle. The first is that the path of a light ray should be bent by gravitational fields, and secondly, there should be a gravitational redshift.

1.1.3 Bending of light in a gravitational field

Consider a uniform gravitational field, g. A lift is in free fall in this gravitational field. Of course from the equivalence principle all bodies that do not experience electromagnetic or other non-gravitational forces should just move in straight lines. Special relativity should hold. Suppose the lift has just started to free fall (see Figure 1), and a photon is emitted horizontally from point X on the left hand side of the lift, and after some time hits the other side. There are two observers of these events. A is in the falling lift, and B stands on the platform outside the lift. Free falling A experiences no gravitational field. B on the other hand experiences the gravitational field.

Figure 1



According to A light travels in a straight horizontal line, and so must hit the far side of the lift at point Y at exactly the same height as point X, where it was emitted. The time taken for the photon to travel the width, L, of the lift must be L/c. Both observers should agree approximately on this time. According to B, who sees the lift accelerating to a speed of $g\,L/c$ at the time the photon hits the far side, the point Y will in fact have moved a vertical distance of $\frac{1}{2}gt^2 = \frac{1}{2}g\,(L/c)^2$. Since A's observation must be correct from the equivalence principle, B can only reconcile matters by accepting that the gravitational field has bent the light path.

(In GRII we shall derive this result rigorously for light deflection in the Schwarzschild metric, which corresponds to the exterior spacetime induced by a spherical mass. The deflection of light was one of the classical tests of GR.)

Exercise: Calculate the angular deflection of light at the surface of the Earth. Consider a horizontal path of 1 km length.

1.1.4 Gravitational redshift of spectral lines

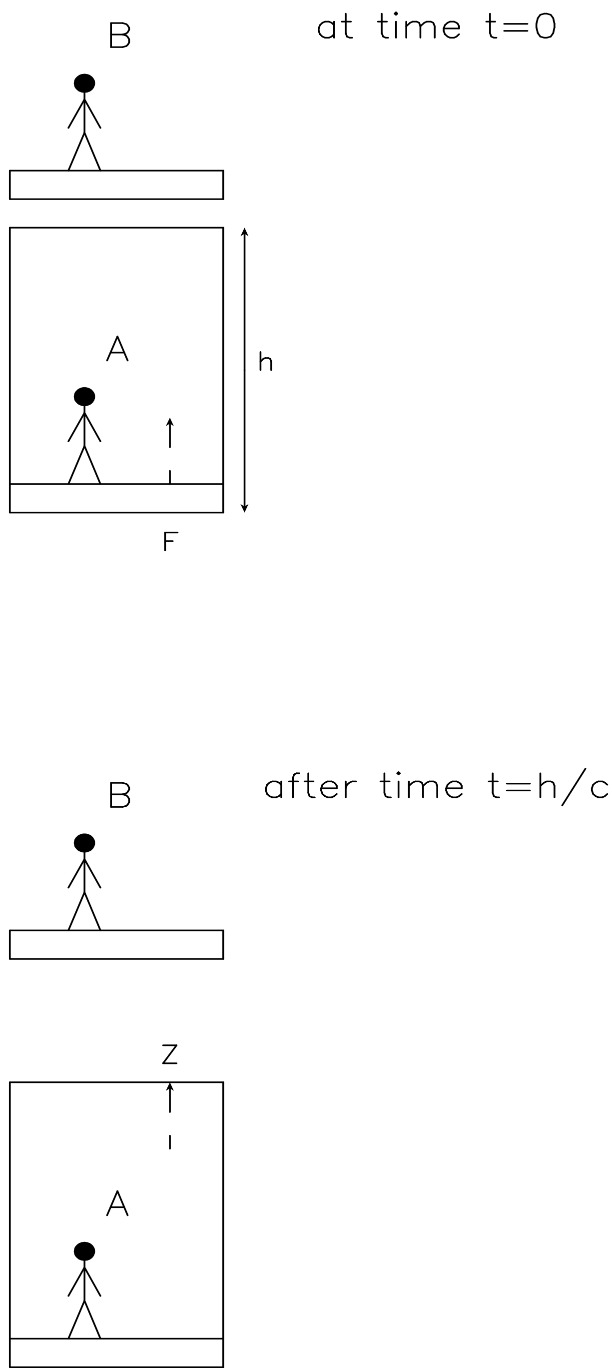
With a similar set up as before, now consider a photon emitted upward from the floor of the lift at point F just at the time the lift is allowed to free fall, and that strikes a detector on the ceiling at point Z (see Figure 2). This time our two observers are A, inside the lift, and B, on a platform above the lift. What frequency does A observe when the photon strikes the detector? The photon must have the same frequency as when emitted, because A is locally inertial. If B now conducts the experiment and measures the frequency of the arriving photon, what does B see? According to A, for whom everything is in agreement with special relativity, observer B is receding at speed $v = g\ h/c$ when the photon reaches F. So B would observe the photon to have a redshift of $v/c = g\ h/c^2$. A of course says this is simply a Doppler shift, since B is moving away from the source. B must attribute the shift to the gravitational field – in "climbing" out of the gravitational field, the photon "loses" energy, and so is redshifted.

This redshift can also be expressed in terms of the change in gravitational potential, ϕ, since $gh = -\delta\phi$. Thus $\delta\lambda/\lambda = -\delta\phi/c^2$. Emerging from the gravitational field the photon will be observed to be redshifted. This effect has been observed in the spectral lines of white dwarf stars. On the Earth both this and the bending of light are very weak effects.

Exercise: Calculate the gravitational redshift of a photon moving upward through 1 km vertical height at the Earth's surface.

Exercise: What is the gravitational redshift of a spectral line emitted at the surface of a typical white dwarf star and observed at the Earth?

Figure 2



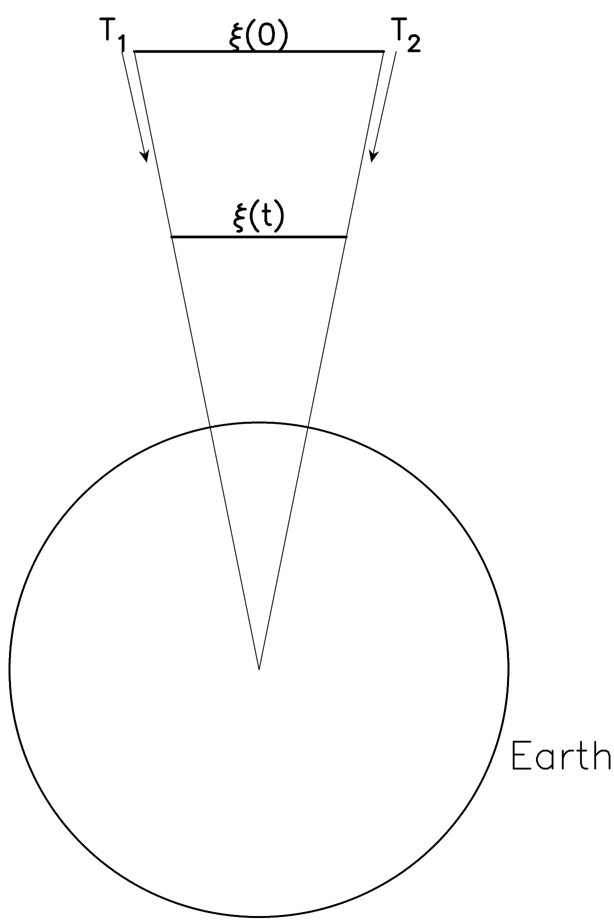
1.2 Spacetime curvature

As we have already mentioned, gravitation appears in GR as spacetime curvature. Let us see how this arises.

1.2.1 Locally inertial frames (LIF)

So far we have considered only uniform gravitational fields. Generally gravitational fields are not uniform. Thus a free falling frame is only inertial over a limited spatial and temporal region around a given event. Consider for instance two free test particles that are separated by a small distance and initially at rest with respect to a LIF – see Figure 3. The fact that this is only a locally inertial frame is reflected by the fact that the distance between the two test particles will noticeably change after a certain period of time.

Figure 3



Exercise: Take the situation above the Earth's surface. The initial separation of the two test particles, T_1 and T_2 is ξ_0. Take $\xi_0 = 25$ m. After 7s suppose the separation of T_1 and T_2 is $\xi = \xi_0 + \Delta\xi_0$. Show that $\Delta\xi_0 \approx -10^{-3}$m (take $g = 10$ ms^{-2} and the radius, R, of the Earth to be 6×10^6 m. (Hint: $\Delta\xi/\xi = \Delta r/R$, where Δr is the change in the radial distance of the test particles from the Earth's centre.)

1.2.2 Geodesic deviation

The separation, ξ, between the two free test particles is called the **geodesic deviation**. In general ξ is a vector. (In fact, it is a **four vector**, if we consider time separations as well – see later). It is the acceleration of this geodesic deviation that indicates the presence of a gravitational field, or, as we shall see later, the curvature of spacetime. In the simple example illustrated in Figure 3, we can define our coordinate system so that only the x (i.e. horizontal) component of ξ is non-zero. We denote this component by ξ_x. From similar triangles we have

$$\frac{\xi_x + \Delta\xi_x}{r + \Delta r} = \frac{\xi_x}{r} = k \tag{4}$$

where k is a constant. Taking derivatives with respect to time gives

$$\ddot{\xi}_x = k\ddot{r} = -\frac{kGM}{r^2} \tag{5}$$

Substituting for $k = \xi_x/r$ yields

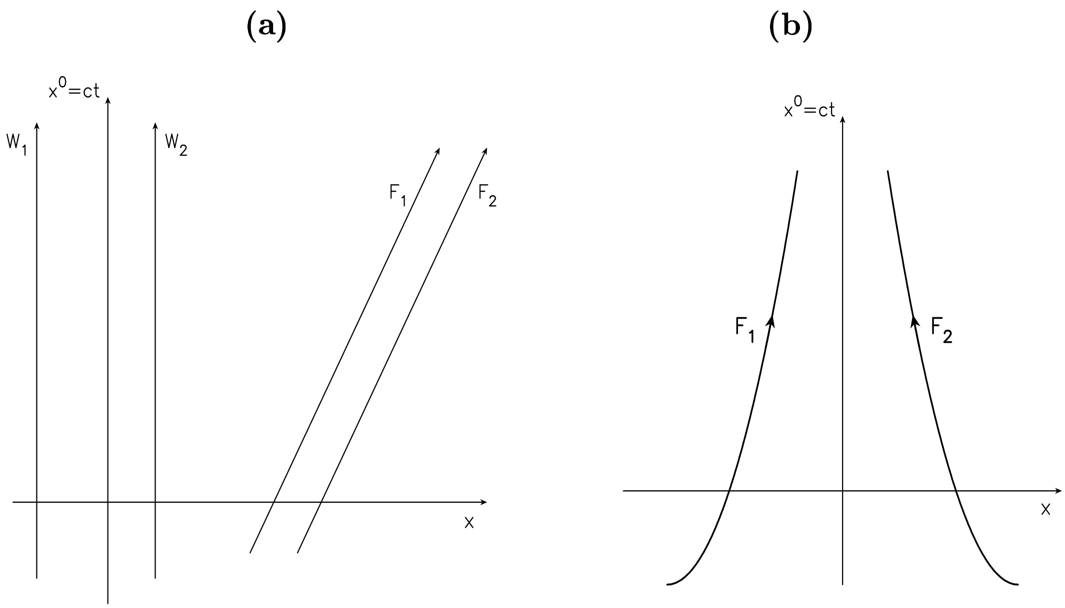
$$\ddot{\xi}_x = -\frac{\xi_x}{r}\frac{GM}{r^2} = -\frac{GM\xi_x}{r^3} \tag{6}$$

At the Earth's surface, $r \approx R$ so $\ddot{\xi}_x = -GM\xi_x/R^3$. Rewriting in more sensible units we obtain

$$\frac{d^2\xi_x}{d(ct)^2} = -\frac{GM}{R^3c^2}\,\xi_x \tag{7}$$

In the flat spacetime of Minkowski free test particles have worldlines that are 'straight'. Thus the acceleration of the geodesic deviation is zero for Minkowski spacetime – see Figure 4a. Worldlines W_1 and W_2 remain parallel, as do worldlines F_1 and F_2.

Figure 4



Test particles in GR have worldlines that are geodesics, but now – because of the presence of matter – the spacetime is **not** flat. Consequently the geodesics are not 'straight lines', and there is an acceleration of the geodesic deviation – see Figure 4b. Worldlines F_1 and F_2 do not remain parallel. In the example illustrated in Figure 3, the test particles are initially at rest with respect to the LIF, but almost imperceptibly they move towards each other as they fall towards Earth. Notice that in our equation

$$\frac{d^2\xi_x}{d(ct)^2} = -\frac{GM}{R^3c^2}\ \xi_x \tag{8}$$

the factor

$$\frac{GM}{R^3c^2}$$

has the dimensions m^{-2}. Evaluated at the Earth's surface this quantity has the value $10^{-23}\ \mathrm{m}^{-2}$.

Exercise: Evaluate the factor

$$\frac{GM}{R^3c^2}$$

at the surface of the Earth, the Sun, and a neutron star of one solar mass.

We can understand equation (8) in terms of a 2-D analogy. Suppose T_1 and T_2 are on the equator of a sphere of radius a (see Figure 5). Consider geodesics perpendicular to the equator passing through T_1 and T_2 . The arc distance along the geodesics is denoted by s and the separation of the geodesics at s is $\xi(s)$. Evidently this geodesic separation is not constant as we change s. Let us write down the differential equation governing the acceleration of this geodesic separation or deviation. If $\xi(0)$ is the initial deviation, we may write $d\phi = \xi(0)/a$ and so

$$\xi(s) = a\cos\theta\, d\phi = \xi(0)\cos\theta = \xi(0)\cos s/a \tag{9}$$

Differentiating $\xi(s)$ twice with respect to s yields

$$\frac{d^2\xi}{ds^2} = -\frac{1}{a^2}\xi \tag{10}$$

Compare this with equation (8). In some sense the quantity

$$\mathcal{R} = \left\{\frac{GM}{R^3c^2}\right\}^{-\frac{1}{2}}$$

represents the radius of curvature of spacetime at the surface of the Earth.

Exercise: Sketch on Figure 6 the worldline of the Earth, taking the sun as the origin of the coordinate system.

Figure 5

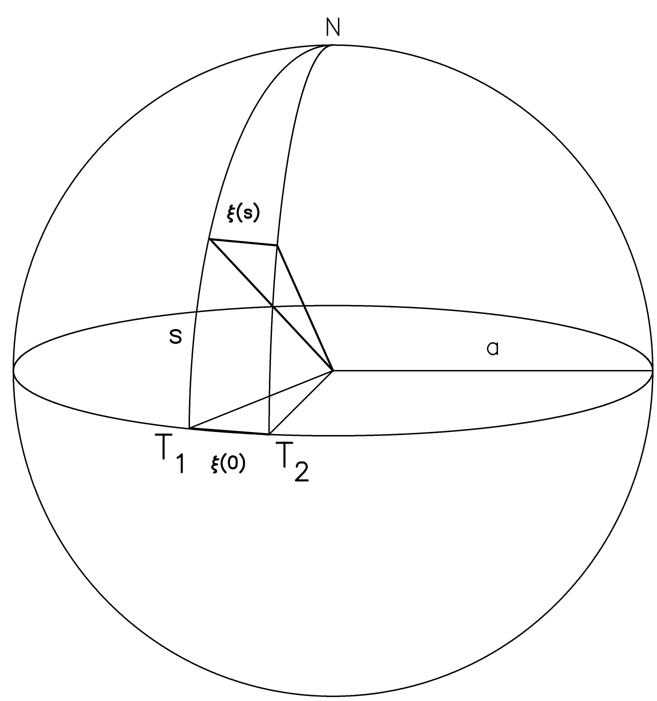


Figure 6