Astronomy A3/A4H, Physics P4H

Gravitation and Relativity I: Example Sheet 6

1. From the definition of the Riemann Christoffel tensor in its rank $(1,3)$ form, given in equation (246), show that it is *skew-symmetric* in its second and third lower indices, i.e.

$$R^{\mu}{}_{\alpha\beta\gamma} = -R^{\mu}{}_{\alpha\gamma\beta}$$

Hence show that, for all μ, α, β

$$R^{\mu}{}_{\alpha\beta\beta} = 0$$

2. Using equation (251) for the fully covariant form of the Riemann Christoffel tensor, viz

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma})$$

verify the following symmetry relations for $R_{\alpha\beta\gamma\delta}$:-

$$\begin{aligned} R_{\alpha\beta\delta\gamma} &= -R_{\alpha\beta\gamma\delta} \\ R_{\beta\alpha\gamma\delta} &= -R_{\alpha\beta\gamma\delta} \\ R_{\gamma\delta\alpha\beta} &= R_{\alpha\beta\gamma\delta} \\ R_{\beta\alpha\delta\gamma} &= R_{\alpha\beta\gamma\delta} \end{aligned}$$

Thus verify that

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

3. Using the symmetry relations derived in Q.2 for the fully covariant form of the Riemann Christoffel tensor, show that the Ricci tensor

$$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma} = g^{\sigma\delta} R_{\sigma\alpha\delta\gamma}$$

is symmetric, i.e. $R_{\alpha\gamma} = R_{\gamma\alpha}$. Show that the contravariant Ricci tensor, $R^{\alpha\gamma}$, is also symmetric.

4. Explain why the equation

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = kT^{\mu\nu} + \Lambda g^{\mu\nu}$$

where Λ is a constant, is a possible alternative solution to Einstein's equations. (We call Λ the *cosmological constant*).

5. Verify equation (281), i.e. $h^{\nu\gamma} = -\eta^{\alpha\gamma}\eta^{\nu\beta}h_{\alpha\beta}$, to first order in $h_{\alpha\beta}$, where the terms are as defined by equations (278) - (280).

6. An *orthogonal* metric is one for which the coefficients, $g_{\mu\nu} = 0$ if $\mu \neq \nu$. Show that for an orthogonal metric, the Christoffel symbols take the form:-

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} &= 0 \quad \text{for } \lambda,\ \mu,\ \nu \text{ all different} \\ \Gamma^{\lambda}_{\lambda\mu} &= \Gamma^{\lambda}_{\mu\lambda} = g_{\lambda\lambda,\mu}/2g_{\lambda\lambda} \\ \Gamma^{\lambda}_{\mu\mu} &= -g_{\mu\mu,\lambda}/2g_{\lambda\lambda} \\ \Gamma^{\lambda}_{\lambda\lambda} &= g_{\lambda\lambda,\lambda}/2g_{\lambda\lambda} \end{aligned}$$

7. Consider the line element for $\mathbf{S^2}$, a sphere of radius, a

$$ds^2 = a^2(d\theta^2 + sin^2\theta d\phi^2)$$

(a) Write down the components of the metric tensor, and the Christoffel symbols in this coordinate system. (Hint, see Tutorial sheet 3, Q.5)

(b) Use the Christoffel symbols to compute all 16 components of the Riemann Christoffel tensor in polar coordinates. (Hint, only four of these components are independent, from the symmetries of the Riemann Christoffel tensor).

(c) Hence compute the four components of the Ricci tensor and the curvature scalar.

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