Astronomy A3/A4H, Physics P4H

Gravitation and Relativity I: Example Sheet 5

1. The line element for the unit sphere, $\mathbf{S^2}$, embedded in $\mathbf{E^3}$ is given by

$$dl^2 = d\theta^2 + sin^2\theta d\phi^2$$

Write down the Christoffel symbols for this metric. (Hint: these follow trivially from the results of Tutorial Sheet 3, Q. 5). Given that the geodesic equation is

$$\frac{d^2x^\mu}{dl^2} + \Gamma^\mu_\alpha\beta\frac{dx^\alpha}{dl}\frac{dx^\beta}{dl} = 0$$

where l is an affine parameter, show that the curve $\theta = l$, $\phi =$ constant is a geodesic.

2. Let $T^\mu = dx^\mu/d\eta$ be a tangent vector to a geodesic curve with parameter, η. Recall from the lecture notes equation (179), which gives the geodesic equation in terms of the scalar function, $f(\eta)$, viz

$$f(\eta)\frac{dT^\mu}{d\eta} + T^\mu\frac{df}{d\eta} + f(\eta)\Gamma^\mu_\alpha\beta T^\alpha(\eta)T^\beta(\eta) = 0$$

Consider the affine transformation, introduced in the lecture notes

$$\lambda = -C_0\int f(\eta)^{-1}d\eta + \lambda_0$$

Show that

$$f(\eta) = C_0\left(\frac{d\lambda}{d\eta}\right)^{-1} \quad \text{and} \quad \frac{df}{d\eta} = C_0\frac{d^2\lambda}{d\eta^2}\left(\frac{d\lambda}{d\eta}\right)^{-2}$$

Hence verify that the above geodesic equation reduces to equation (182) of the lecture notes, i.e.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_\alpha\beta\frac{dx^\alpha}{d\lambda}\frac{d\beta}{d\lambda} = 0$$

3. If θ and ϕ are the co-latitude and longitude respectively on the surface of the unit sphere, $\mathbf{S^2}$, Mercator's projection is obtained by plotting (x, y) as rectangular Cartesian coordinates in $\mathbf{E^2}$, where

$$x = \phi \qquad y = \log\cot\frac{1}{2}\theta$$

Show that the line element for $\mathbf{S^2}$ in terms of x and y is given by

$$ds^2 = \text{sech}^2 y(dx^2 + dy^2)$$

Comment on the comparison of this expression with the line element for $\mathbf{E^2}$ in Cartesian coordinates.

(N.B. This question is a neat mathematical exercise, but don't worry if you can't derive the required result; the algebra is messier than anything you will meet in the class exam or degree exam)

4. Let A_i and B_j be the components of two arbitrary one-forms. Show explicitly that the *product rule* holds for the covariant derivatives of A_i and B_j, i.e.

$$(A_i B_j)_{;k} = A_i(B_{j;k}) + (A_{i;k})B_j$$

Argue that the product rule will hold for the covariant derivatives of the product of two or more tensors of arbitrary type

5. Recall that the covariant components of a vector with contravariant components, V^β, are

$$V_\alpha = g_{\alpha\beta} V^\beta$$

By taking the covariant derivative of this equation, show that

$$g_{\alpha\beta;\gamma} = 0$$

(Hint: recall the result of Tutorial Sheet 4, Q. 7)

6. By noting that the equation $g_{\alpha\beta;\gamma} = 0$ is a tensor equation, use the principle of equivalence to establish that the equation is valid in any reference frame. Deduce also that $g^{\alpha\beta}_{\;\;;\gamma} = 0$

7. By direct differentiation of the expression

$$g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

with respect to the proper time, τ, show that the magnitude of the tangent vector, $dx^\alpha/d\tau$, is constant along the world-line of a material particle.

(Hint: remember this is a tensor equation, and so must hold in any frame)

8. In the MCRF of a fluid element, the components, u^μ, of the four velocity are identically zero. Why, in general, can we not assume that $u^\mu_{\;,\nu} = 0$?

9. In the energy momentum tensor, the component T^{0i} represents the energy flux through a surface of constant x^i, while the component T^{i0} represents the density of the i^{th} component of momentum. Suppose we have a fluid element, with spatial dimensions Δx, Δy and Δz parallel to the $x^1 \equiv x$, $x^2 \equiv y$ and $x^3 \equiv z$ axes, containing particles of equal rest mass, m. By considering a stream of particles moving with 3-velocity, v, in the positive x direction, explain why it follows that $T^{01} = T^{10}$, with similar arguments giving $T^{0i} = T^{i0}$, for $i = 1, 2, 3$

10. In an ideal gas we have a collection of non-colliding particles with an isotropic, random distribution of velocities. Consider a fluid element containing an ideal gas in which all the particles have rest mass, m and speed, v, in the MCRF. Further, let n denote the number density of gas particles in the MCRF. Explain why, given these

assumptions, the energy momentum tensor for an ideal gas takes the same form as that of a perfect fluid, i.e.

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Let the fluid element have spatial dimensions Δx, Δy and Δz, parallel to the x, y and z axes in the MCRF. Consider those particles in the fluid element moving with 3-velocity, v, (where $v << c$) in a direction which makes an angle, θ, with the normal to the $y - z$ plane. Show that the x component of momentum transferred in the x direction from these particles in time Δt is

$$\Delta p = mv^2 \cos^2\theta \; n \Delta y \Delta z \Delta t$$

and hence that the x component of momentum flux through the $y - z$ plane is

$$T^{xx} = \frac{1}{3}\rho v^2$$

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