Astronomy A3/A4H, Physics P4H

Gravitation and Relativity I: Example Sheet 4

1. Let A_j be the components of an arbitrary one-form. Write down the transformation law for A_j, and for its covariant derivative, $A_{j;k}$. By considering the expression for $A_{j;k}$ in a primed coordinate system, show that the transformation law for the Christoffel symbols takes the form of equation (147), i.e.

$$\Gamma'^{i}_{jk} = \frac{\partial x'^i}{\partial x^r}\frac{\partial x^s}{\partial x'^j}\frac{\partial x^t}{\partial x'^k}\Gamma^r_{st} + \frac{\partial x'^i}{\partial x^l}\frac{\partial^2 x^l}{\partial x'^j x'^k}$$

2. Suppose that in one coordinate system the Christoffel symbols are symmetric in their lower indices, i.e. $\Gamma^i_{jk} = \Gamma^i_{kj}$. By considering the transformation law for the Christoffel symbols, show that they will be symmetric in any coordinate system.

3. Let ϕ be a scalar field. Recall from your notes that $\phi_{,i}$ transforms as a $(0,1)$ tensor. Hence show that

$$\phi_{,i;k} - \phi_{,k;i} = \phi_{,ik} - \phi_{,ki}$$

provided that the Christoffel symbols are symmetric in their lower indices.

4. Let Γ^i_{jk} and Γ^{*i}_{jk} denote the components of two different affine connections, in a given coordinate system, which both transform according to equation (147). Show that the difference

$$\Upsilon^i_{jk} = \Gamma^i_{jk} - \Gamma^{*i}_{jk}$$

transforms as a tensor. What is the (m,n) type of this tensor?

5. Consider the vector field $\vec{V}$ with Cartesian components $\{x^2 + 3y, y^2 + 3x\}$

   (a) Using the transformation law for a $(1,0)$ tensor, and the results of Q.1 of Sheet 2, determine $\{V^r, V^\theta\}$, the components of $\vec{V}$ with respect to the polar coordinate basis, $\{\vec{e}_r, \vec{e}_\theta\}$.

   (b) Write down the components of the covariant derivative, $V^i_{;j}$ in Cartesian coordinates.

   (c) Using the fact that $V^i_{;j}$ transforms as a $(1,1)$ tensor, compute the components of the covariant derivative in the polar coordinate system.

   (d) Now compute directly the polar components of the covariant derivative of $\vec{V}$, using the definition of the covariant derivative in terms of the partial derivatives of the vector components and the Christoffel symbols. (Hint: you can use the results given in equations (143) - (146) which calculate the Christoffel symbols for polar coordinates).

   (e) Verify that the polar components computed in parts (c) and (d) are in agreement.

6. Consider now the one-form field $\tilde{A}$ with Cartesian components $\{x^2 + 3y, y^2 + 3x\}$

   (a) Using the transformation law for a $(0, 1)$ tensor, and again the results of Q.1 of sheet 2, determine $\{A_r, A_\theta\}$, the components of $\tilde{A}$ with respect to the polar coordinate basis.

   (b) Compute the components of the covariant derivative, $A_{i;j}$ in Cartesian coordinates.

   (c) Using the fact that $A_{i;j}$ transforms as a $(0, 2)$ tensor, compute the components of the covariant derivative in the polar coordinate system.

   (d) Now compute directly the polar components of the covariant derivative of $\tilde{A}$, using the definition of the covariant derivative in terms of the partial derivatives of the one-form components and the Christoffel symbols.

   (e) Again, verify the agreement of the polar components computed in parts (c) and (d).

7. Considering your results from Q.5 and Q.6, verify that in both Cartesian and polar coordinates

$$g_{ik} V^k_{;j} = A_{i;j}$$

Dr. Martin Hendry
Room 607, Kelvin Building
Ext 5685; email martin@astro.gla.ac.uk