# Astronomy A3/A4H, Physics P4H

## Gravitation and Relativity I: Example Sheet 3

1. Write down the $(m, n)$ type of the following tensors:-

$$(a)\, A^{ijkl}_{mn} \quad (b)\, B^{ijmn}_{mn} \quad (c)\, B^{ijkl}_{mnp}$$

2. Tensor components $A^{ij}$ and $B^{ij}$ are equal in one coordinate frame. By considering the tranformation law for a $(2,0)$ tensor, or otherwise, show that they must be equal in any coordinate frame. Show that if $A^{ij}$ is symmetric in one coordinate frame it is symmetric in any frame.

3. If $T^{ij}$ is a $(2,0)$ tensor, show that so too are $S^{ij} = T^{ij} + T^{ji}$ and $A^{ij} = T^{ij} - T^{ji}$

4. Show that if $A_i$ is a $(0,1)$ tensor, then $A_{i,j} - A_{j,i}$ is a $(0,2)$ tensor. We call $A_{i,j} - A_{j,i}$ the *curl* of $\tilde{A}$. If $\tilde{A}$ is the gradient of a scalar field, $\phi$, show that curl $(\tilde{A})$ vanishes in any coordinate frame.

5. The line element in $\mathbf{E^3}$ expressed in Cartesian coordinates $\{x, y, z\}$ is given by

$$dl^2 = dx^2 + dy^2 + dz^2$$

   Write down the metric tensor components, $g_{xx}$, $g_{yy}$ etc.

   Now consider spherical polar coordinates $\{r, \theta, \phi\}$ in $\mathbf{E^3}$. Write down the line element and metric tensor components in terms of this coordinate system. Writing

$$\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\vec{k}$$

   express the basis vectors $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi\}$ in terms of the Cartesian basis $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$.

   Finally, calculate the Christoffel symbols in both Cartesian and polar coordinates using the equation

$$\frac{\partial \vec{e}_i}{\partial x^j} = \Gamma^k_{ij}\, \vec{e}_k$$

6. Given that the Christoffel symbols for a Riemannian space may be written as

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( g_{lj,k} + g_{lk,j} - g_{jk,l} \right)$$

   verify the results of Q.5 by direct differentiation of the metric tensor components

7. The line element in $\mathbf{E^2}$ expressed in Cartesian coordinates $\{x, y\}$ is

$$dl^2 = dx^2 + dy^2$$

A new coordinate system, $\{u, v\}$, is defined by

$$u = \frac{1}{2}(x^2 - y^2) \quad \text{and} \quad v = xy$$

(a) Show that the line element in these new coordinates is

$$dl^2 = \frac{1}{2}(u^2 + v^2)^{-\frac{1}{2}}(du^2 + dv^2)$$

(b) A one-form has Cartesian components $\{A_x, A_y\}$ and components $\{A_u, A_v\}$ in the new coordinate system. Show that

$$A_u = \frac{(xA_x - yA_y)}{(x^2 + y^2)}$$

and derive the corresponding expression for $A_v$.

Dr. Martin Hendry
Room 607, Kelvin Building
Ext 5685; email martin@astro.gla.ac.uk