Astronomy A3/A4H, Physics P4H

Gravitation and Relativity I: Example Sheet 2

1. (a) Write down the transformation law from circular polar coordinates, $\{r, \theta\}$, to Cartesian coordinates, $\{x, y\}$; i.e. determine the functions

$$\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}$$

(b) By differentiating the equations

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \text{atan}\left(\frac{y}{x}\right)$$

determine the functions

$$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$$

which define the inverse transformation from Cartesian to polar coordinates. Hence, form the Jacobian matrices of these two transformations and verify that the product of these two Jacobians is the identity matrix.

(c) Let $\vec{v}$ be the vector with components $\{x, y\}$ with respect to the Cartesian basis, i.e.

$$\vec{v} = x\,\vec{e}_x + y\,\vec{e}_y$$

Show that $\dot{\vec{v}}$ and $\ddot{\vec{v}}$ have components $\{\dot{x}, \dot{y}\}$ and $\{\ddot{x}, \ddot{y}\}$ with respect to this basis.

(d) Using the relations $x = r\cos\theta$, $y = r\sin\theta$, write down expressions for $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ in terms of polar coordinates r and θ and their time derivatives

(e) Now use the general transformation law for the components of a vector

$$v'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} v^{\nu}$$

to express the components of $\dot{\vec{v}}$ and $\ddot{\vec{v}}$ with respect to the *polar* basis, $\{\vec{e}_r, \vec{e}_\theta\}$. Hence show that

$$\dot{\vec{v}} = \dot{r}\,\vec{e}_r + \dot{\theta}\,\vec{e}_\theta$$

$$\ddot{\vec{v}} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

2. Let the scalar field, ϕ, defined on $\mathbf{E^2}$, be

$$\phi(x, y) = x^2 + y^2 + 2xy$$

for Cartesian coordinates $\{x, y\}$.

(a) Write down the components of the gradient one-form, $\tilde{d\phi}$, with respect to the Cartesian basis.

(b) Using the general transformation law for the components of a one-form

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$$

determine the components of $\tilde{d\phi}$ in terms of polar coordinates $\{r, \theta\}$. (Hint: you will need to use the results of Q. 1(a) to define the transformation law)

(c) By expressing ϕ in terms of r and θ, obtain directly the polar components of $\tilde{d\phi}$ and verify that they agree with those obtained in (b)

(d) From the $\mathbf{E^2}$ line element in Cartesian coordinates, determine the components of the contravariant metric tensor, i.e. g^{xx}, g^{xy}, g^{yx}, g^{yy}. Hence determine the Cartesian components of the *vector* gradient, $\vec{d\phi}$, (i.e. with raised index)

(e) Now, from the line element in polar coordinates, determine g^{rr}, $g^{r\theta}$, $g^{\theta r}$ and $g^{\theta\theta}$. Hence determine the polar components of $\vec{d\phi}$. Comment on your answers to part (d) and (e)

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