Astronomy A3/A4H, Physics P4H

Gravitation and Relativity I: Example Sheet 1

1. In an inertial frame Newton's law may be written as

$$F^i = m\frac{d^2x^i}{dt^2}$$

Show that Newton's law still holds under a coordinate transformation

$$x'^i = R^i_j x^j + v^i t + c^i$$

(where R^i_j, v^i and c^i are time-independent) provided that $F'^i = R^i_j F^j$

2. For the transformations

$$x'^i = R_1{}^i_j x^j + v_1{}^i t + c_1^i$$

$$x''^i = R_2{}^i_j x'^j + v_2{}^i t + c_2^i$$

show that

$$x''^i = S^i_j x^j + w^i t + a^i$$

where $S^i_j = R_2{}^i_k R_1{}^k_j$, $w^i = R_2{}^i_k v_1{}^k + v_2{}^i$ and $a^i = R_2{}^i_j c_1{}^j + c_2^i$

Comment on your answer.

3. Show that, for any μ, ν

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

and that

$$\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\nu} = \delta^\mu_\nu$$

Hence, or otherwise, show that the components of the Kronecker delta are identical in any coordinate system, i.e.

$$\delta'^\mu_\nu = \delta^\mu_\nu$$

4. The basis vectors, $\{\vec{e'_1}, \vec{e'_2}\}$, of a rotating coordinate system, $\{x'^1, x'^2\}$, are given by

$$\vec{e'_1} = \cos\omega t\,\vec{e_1} + \sin\omega t\,\vec{e_2}$$

$$\vec{e'_2} = -\sin\omega t\,\vec{e_1} + \cos\omega t\,\vec{e_2}$$

where $\{\vec{e_1}, \vec{e_2}\}$ are the basis vectors of an inertial, Cartesian system of coordinates $\{x^1, x^2\}$.

Show that

$$\frac{d\vec{e'_1}}{dt} = \omega \vec{e'_2} \quad \text{and} \quad \frac{d\vec{e'_2}}{dt} = -\omega \vec{e'_1}$$

Further, show that we can write these two equations in the form

$$\frac{d\vec{e'_i}}{dt} = \Omega_i^j \vec{e'_j}$$

where $\Omega_1^1 = \Omega_2^2 = 0$ and $\Omega_1^2 = -\Omega_2^1 = \omega$

The displacement vector of a particle is $\vec{x} = x^i \vec{e_i} = x'^i \vec{e'_i}$. By differentiating this expression, show that the velocity vector may be written as

$$\vec{v} = \frac{d\vec{x}}{dt} = \left(\frac{dx'^i}{dt} + x'^j \Omega_j^i \right) \vec{e'_i}$$

By differentiating again, write down an expression for the acceleration vector in terms of the components in the rotating coordinate system. Compare your answer with the result for the Cartesian frame.

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