Galaxies II Examples 6 Model Answers

1) (a) I.

$$\frac{N_{2-3}}{N_{10-11}} = \int_2^3 M^{-2.35}\,dM \Big/ \int_{10}^{11} M^{-2.35}\,dM$$

$$= \left[\frac{M^{-1.35}}{1.35}\right]_3^2 \Big/ \left[\frac{M^{-1.35}}{1.35}\right]_{11}^{10}$$

$$= 0.16536 \big/ 5.393\times10^{-3} \simeq 30.7$$

II.

$$\frac{N_{2-3}}{N_{10-11}} = \left[M^{-0.8}\right]_3^2 \Big/ \left[M^{-0.8}\right]_{11}^{10}$$

$$= 0.1591 \big/ 1.164\times10^{-2} \simeq 13.7$$

(b)

$$\phi(L)\,dL \propto \xi(M)\,dM = A\,M^{-(1+x)}\,dM$$

$$\Rightarrow \quad \phi(L) = A\,M^{-(1+x)}\frac{dM}{dL}$$

$$M = M_\odot\left(\frac{L}{L_\odot}\right)^{1/\alpha} \qquad \frac{dM}{dL} = \frac{M_\odot}{\alpha L_\odot}\left(\frac{L}{L_\odot}\right)^{\frac{1}{\alpha}-1}$$

$$\therefore \quad \phi(L) = \frac{A M_\odot}{\alpha L_\odot} M_\odot^{-(1+x)}\left(\frac{L}{L_\odot}\right)^{-\frac{(1+x)}{\alpha}}\left(\frac{L}{L_\odot}\right)^{\frac{1}{\alpha}-1}$$

$$= \frac{A M_\odot^{-x}}{\alpha L_\odot}\left(\frac{L}{L_\odot}\right)^{-\left(\frac{x+\alpha}{\alpha}\right)}$$

(c)

$$\frac{N_{2-3}}{N_{10-11}} = \int_2^3 \phi(L)\,dL \Big/ \int_{10}^{11}\phi(L)\,dL$$

take $x = 0.8$, $\alpha = 3.5$

$$= \left[\left(L/L_\odot\right)^{-0.228}\right]_3^2 \Big/ \left[\left(L/L_\odot\right)^{-0.228}\right]_{11}^{10}$$

$$= 5.9$$

d) This ratio is smaller than in part (a), since stars with $2L_\odot < L < 3L_\odot$ and $10L_\odot < L < 11L_\odot$ have mass ranges closer to $M_\odot$

(e.g. for $L/L_\odot = 10$, $M/M_\odot \simeq 1.93$)

2) (a) τ_{MS} for $M = 2M_\odot$ is 1.1 Gyr

=> to still be on MS, must have been formed within last 1.1 Gyr

$$\Rightarrow f_{MS} = \int_{8.9}^{10} \psi(t)\,dt \Big/ \int_0^{10} \psi(t)\,dt$$

$$= \int_{8.9}^{10} e^{-t/\tau}\,dt \Big/ \int_0^{10} e^{-t/\tau}\,dt$$

$$= \left[-\tau e^{-t/\tau}\right]_{8.9}^{10} \Big/ \left[-\tau e^{-t/\tau}\right]_0^{10}$$

$$= \left(e^{-8.9/3} - e^{-10/3}\right) \Big/ \left(1 - e^{-10/3}\right) = 0.016$$

(b) We want $A e^{-10/3} = 5.0\ M_\odot\ pc^{-2}\ Gyr^{-1}$

$$\Rightarrow A = 140\ M_\odot\ pc^{-2}\ Gyr^{-1}$$

c) $$\Psi_{INT} = \int_0^{10} A e^{-t/\tau}\,d\tau = \left[-\tau A e^{-t/\tau}\right]_0^{10}$$

$$= \tau A\left[1 - e^{-10/3}\right]$$

$$= 3 \times 140 \times \left[1 - 0.036\right]$$

$$= 405\ M_\odot\ pc^{-2}$$

d) total (past + future) $= 420\ M_\odot\ pc^{-2}$

=> fraction to date = 0.964

(e) Some weaknesses : monotic SFR – ignores starburst episodes, due to e.g. mergers ; assumes constant SFR everywhere ; no account of chemical evolution / recycling

3) a) total mass of stars $= 5.0\, M_\odot \times \pi \times (15000)^2 \times 1\text{ Gyr}$

$$= 3.53 \times 10^9\, M_\odot$$

(b) total mass $= C \int_{0.1}^{50} M\, \xi(M)\, dM$ (M in solar masses)

$$= C \int_{0.1}^{50} M^{-1.35}\, dM$$

$$= C \left[\frac{M^{-0.35}}{-0.35} \right]_{0.1}^{50}$$

$$= \frac{C}{0.35} \left[0.1^{-0.35} - 50^{-0.35} \right]$$

we must be able to write $C = K^{1.35}$, where K is in solar masses, to ensure the correct dimensions

$\Rightarrow$ total mass $= \dfrac{K^{1.35}}{0.35} \left[0.1^{-0.35} - 50^{-0.35} \right]$

c) We require $\dfrac{K^{1.35}}{0.35} \left[0.1^{-0.35} - 50^{-0.35} \right] = 3.53 \times 10^9\, M_\odot$

$$\Rightarrow \quad K^{1.35} = 6.226 \times 10^8$$

$$\Leftrightarrow \quad K = 3.27 \times 10^6\, M_\odot$$

d) Total <u>number</u> of stars $= K^{1.35} \int_{0.1}^{50} M^{-2.35}\, dM$ in 1 Gyr

$$= K^{1.35} \left[\frac{M^{-1.35}}{-1.35} \right]_{0.1}^{50} = 1.03 \times 10^{10} \Rightarrow 10.3 \text{ stars / yr}$$

e) No. per year $(M_\odot \to 2M_\odot) = 6.226 \times 10^8 \left[\frac{M^{-1.35}}{-1.35} \right]_1^2 / 10^9$

$= 0.28$ stars yr^{-1}

No per year $(10M_\odot \to 20M_\odot) = 6.226 \times 10^8 \left[\frac{M^{-1.35}}{-1.35} \right]_{10}^{20} / 10^9$

$= 0.0125$ stars yr^{-1}

4) a) When all the gas is gone $M_s = M_g(0)$

Fraction with metallicity between Z and $Z + dZ$ $= \frac{dM_s}{M_s}$

$$df = \frac{1}{p} \exp\left[-\frac{Z - Z_0}{p} \right] dZ$$ as required

b) $$\langle Z \rangle = \int_{\text{Stars}} Z \, df$$

$$= \frac{1}{p} \int_0^\infty Z e^{-Z/p} \, dZ$$ with $Z_0 = 0$

Put $y = Z/p \Rightarrow Z = p\,dy \quad dZ = p\,dy$

$$\Rightarrow \langle Z \rangle = \frac{p^2}{p} \int_0^\infty y e^{-y} \, dy$$

$$= p\left[-y e^{-y} \Big|_0^\infty + \int_0^\infty e^{-y} \, dy \right] = p$$

5) (a) $$\Delta M_s + \Delta M_g = \nu \Delta M_s \Rightarrow \Delta M_s (1-\nu) = -\Delta M_g$$

$$\Delta M_h = Z \Delta M_g + M_g \Delta Z$$

$$= p \Delta M_s - Z \Delta M_s \quad \text{as before}$$

$$\therefore \quad Z \Delta M_g + M_g \Delta Z = \frac{(p - Z) \Delta M_g}{(\nu - 1)}$$

$$\Leftrightarrow \quad \Delta Z = \left(\frac{p - Z}{\nu - 1} - Z \right) \frac{\Delta M_g}{M_g}$$

$$= \frac{p - \nu Z}{\nu - 1} \frac{\Delta M_g}{M_g}$$

(b) $$\frac{dZ}{p - \nu Z} . (\nu - 1) = \frac{dM_g}{M_g}$$

Integrating, $$\ln(p - \nu Z) . \left(\frac{1-\nu}{\nu}\right) = \ln M_g + \text{const}$$

$$\Leftrightarrow \quad (p - \nu Z) = \left[\frac{M_g}{M_g(0)} \right]^{\nu/(1-\nu)}$$

$$\Leftrightarrow \quad Z(t) = \frac{p}{\nu} \left[1 - \left\{ \frac{M_g(t)}{M_g(t=0)} \right\}^{\nu/(1-\nu)} \right] \quad \text{as required}$$

(c) Since no gas enters the system, $M_g(t) \leq M_g(t=0)$. As $\nu > 0$, it follows that $Z \leq p/\nu$ for all t.

6) (a) From your notes (3.15)

$$Z(t) = Z_0 + p \ln\left[\frac{M_g(t=0)}{M_g(t)}\right]$$

$M_g(t) = 13\, M_\odot\, pc^{-2}$

$M_g(t=0) = 35 + 13 = 48\, M_\odot\, pc^{-2}$

$\langle z \rangle \simeq 0.7\, Z_\odot$

$\therefore \quad Z_0 = 0.7 Z_\odot - p \ln\left[\frac{48}{13}\right] \simeq 0.7 Z_\odot - 1.3 p$

(b) $$M\left(Z < Z(t)\right) = M_g(0)\left[1 - e^{-\frac{Z(t) - Z_0}{p}}\right]$$

$$\Rightarrow \frac{1 - \exp\left[-\frac{(0.25 Z_\odot - Z_0)}{p}\right]}{1 - \exp\left[-\frac{(0.7 Z_\odot - Z_0)}{p}\right]} = \frac{33}{132} = 0.25$$

[since all have same mass, can relate $M(<Z)$ to $N(<Z)$]

(c) Substituting from part (a)

$$\frac{1 - \exp\left[-\frac{(0.25 Z_\odot - Z_0)}{p}\right]}{1 - \exp\left[-1.3\right]} = 0.25$$

$$\Leftrightarrow \quad \exp\left[-\frac{(0.25 Z_\odot - Z_0)}{p}\right] = 0.818$$

$$\Leftrightarrow \quad -\frac{(0.25 Z_\odot - Z_0)}{p} = -0.2$$

$$\Leftrightarrow \quad 0.357 Z_0 + 0.464 p = Z_0 + 0.2 p$$

$$\Leftrightarrow \quad Z_0 \simeq 0.41 p$$

(d) From part (a) $\quad 0.41 p = 0.7 Z_\odot - 1.3 p \quad \Rightarrow \quad p \simeq 0.41 Z_\odot$

Hence $\quad Z_0 \simeq (0.41)^2 Z_\odot \simeq 0.17 Z_\odot$

Q5 (b) correction

$$\frac{dZ}{p - \nu Z} \cdot (\nu - 1) = \frac{dM_g}{M_g}$$

Integrating :-

$$\ln(p - \nu Z)\frac{(\nu-1)}{\nu} = \ln M_g + \text{const.}$$

Since $Z = 0$ for $t = 0$ => $\text{const} = \ln p \frac{(\nu-1)}{\nu} - \ln(M_g(t=0))$

$$\Rightarrow \ln(p - \nu Z)\frac{(\nu-1)}{\nu} = \ln\left[\frac{M_g}{M_g(t=0)}\right] + \ln p\left(\frac{\nu-1}{\nu}\right)$$

$$\Leftrightarrow \frac{(\nu-1)}{\nu}\ln\left[\frac{p - \nu Z}{p}\right] = \ln\left[\frac{M_g}{M_g(t=0)}\right]$$

$$\Leftrightarrow \left[1 - \left(\frac{\nu}{p}\right)Z\right] = \left[\frac{M_g}{M_g(t=0)}\right]^{\left(\nu/\nu-1\right)}$$

$$\Leftrightarrow Z(t) = \frac{p}{\nu}\left\{1 - \left[\frac{M_g(t)}{M_g(t=0)}\right]^{\left(\frac{\nu}{\nu-1}\right)}\right\}$$ as required