Galaxies II Examples 4 Model Answers

1) (a) Re-arranging given formula :-

$$V_{app} = \frac{V \sin\phi}{1 - \frac{V}{c}\cos\phi}$$

$$\Leftrightarrow \quad V \sin\phi = V_{app} - V_{app}\frac{V}{c}\cos\phi$$

$$\Leftrightarrow \quad V\left[\sin\phi + \frac{V_{app}}{c}\cos\phi\right] = V_{app}$$

$$\Leftrightarrow \quad V = V_{app} \Big/ \left[\sin\phi + \frac{V_{app}}{c}\cos\phi\right] \qquad \text{as required}$$

(b) $V \leq c \quad \Leftrightarrow \quad \frac{V}{c} \leq 1$

$$\Leftrightarrow \quad \frac{V_{app}}{c} \Big/ \left[\sin\phi + \frac{V_{app}}{c}\cos\phi\right] \leq 1$$

$$\Leftrightarrow \quad \frac{V_{app}}{c} \leq \sin\phi + \frac{V_{app}}{c}\cos\phi$$

$$\Leftrightarrow \quad \frac{V_{app}}{c} - \frac{V_{app}}{c}\cos\phi \leq \sin\phi$$

$$\Leftrightarrow \quad \frac{V_{app}^2}{c^2} - 2\frac{V_{app}^2}{c^2}\cos\phi + \frac{V_{app}^2}{c^2}\cos^2\phi \leq 1 - \cos^2\phi$$

$$\Leftrightarrow \quad \cos^2\phi\left[\frac{V_{app}^2}{c^2} + 1\right] - 2\frac{V_{app}^2}{c^2}\cos\phi + \frac{V_{app}^2}{c^2} - 1 \leq 0$$

This is a quadratic in $x = \cos\phi$ of the form $ax^2 + bx + c \leq 0$

Roots are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{2\frac{V_{app}^2}{c^2} \pm \sqrt{4\frac{V_{app}^4}{c^4} - 4\left(\frac{V_{app}^2}{c^2}+1\right)\left(\frac{V_{app}^2}{c^2}-1\right)}}{2\left(\frac{V_{app}^2}{c^2}+1\right)}$$

$$\Leftrightarrow \quad x = \frac{\left(2\frac{v_{app}^2}{c^2} \pm 2\right)}{\left(2\frac{v_{app}^2}{c^2} + 2\right)}$$

$$x = \left(\frac{v_{app}^2}{c^2} - 1\right) \Big/ \left(\frac{v_{app}^2}{c^2} + 1\right) \quad \text{and} \quad x = 1$$

Inequality is satisfied between roots since $\frac{v_{app}^2}{c^2} + 1 > 0$

Hence $v \le c$ for $\left(\frac{v_{app}^2}{c^2} - 1\right) \Big/ \left(\frac{v_{app}^2}{c^2} + 1\right) \le \cos\phi \le 1$

(c) $$\beta = \beta_{app} \Big/ \left[\sin\phi + \beta_{app}\cos\phi\right]$$

$$\frac{\partial\beta}{\partial\phi} = \frac{-\beta_{app}}{\left[\sin\phi + \beta_{app}\cos\phi\right]^2}\left[\cos\phi - \beta_{app}\sin\phi\right]$$

$$\frac{\partial^2\beta}{\partial\phi^2} = \frac{-\beta_{app}}{\left[\sin\phi + \beta_{app}\cos\phi\right]^2}\left[-\sin\phi - \beta_{app}\cos\phi\right] + \frac{2\beta_{app}\left[\cos\phi - \beta_{app}\sin\phi\right]}{\left[\sin\phi + \beta_{app}\cos\phi\right]^3}$$

$$= \frac{\beta_{app}}{\left(\sin\phi + \beta_{app}\cos\phi\right)}\left[1 + \frac{2\left(\cos\phi - \beta_{app}\sin\phi\right)}{\left(\sin\phi + \beta_{app}\cos\phi\right)^2}\right]$$

Setting $\frac{\partial\beta}{\partial\phi} = 0 \Rightarrow \beta_{app} = \frac{\cos\phi}{\sin\phi} = \cot\phi$

and $\frac{\partial^2\beta}{\partial\phi^2} > 0 \Rightarrow$ *minimum* for $\phi = \tan^{-1}\left(\frac{c}{v_{app}}\right)$

(d) $$\beta \ge \beta_{min} = \frac{\cot\phi}{\left(\sin\phi + \frac{\cos^2\phi}{\sin\phi}\right)} = \frac{\frac{\cos\phi}{\sin\phi}\cdot\sin\phi}{\left(\sin^2\phi + \cos^2\phi\right)} = \cos\phi$$

Also $\gamma = \left(1 - \beta^2\right)^{-1/2}$

$$\Rightarrow \quad \gamma \geq \gamma_{min} = \left(1 - \beta_{min}^2\right)^{-1/2}$$

$$= \left(1 - \cos^2\phi\right)^{-1/2} = 1/\sin\phi$$

(e) $$\beta \geq \beta_{min} = \cos\phi = \frac{\cot\phi}{\sqrt{\mathrm{cosec}^2\phi}} = \frac{\cot\phi}{\sqrt{1+\cot^2\phi}} = \sqrt{\frac{\beta_{app}^2}{1+\beta_{app}^2}}$$

(f) $$\beta_{app} = 3{\cdot}5 \quad \Rightarrow \quad \beta \geq \sqrt{\frac{3{\cdot}5^2}{1+3{\cdot}5^2}} = 0{\cdot}962$$

i.e. $v \geq 0{\cdot}962c$

$$\phi = \tan^{-1}\left(\frac{1}{3{\cdot}5}\right) = 15{\cdot}9^\circ$$

For graph see plot at end of file

2) a) When $\cos\phi = \beta$, $$\beta_{app} = \frac{\beta\left(1-\beta^2\right)^{1/2}}{\left(1-\beta^2\right)} = \frac{\beta}{\sqrt{1-\beta^2}} = \gamma\beta$$

b) If $\beta_{app} \geq 1 \Rightarrow \dfrac{\beta}{\sqrt{1-\beta^2}} \geq 1$

$$\Leftrightarrow \quad \beta^2 \geq 1 - \beta^2$$

$$\Leftrightarrow \quad \beta^2 \geq \frac{1}{2}$$

$$\Leftrightarrow \quad \beta \geq \frac{1}{\sqrt{2}} \qquad \text{i.e.} \quad v \geq \frac{c}{\sqrt{2}}$$

3) a) From lecture notes, in the rest frame of the jet, arrival times of photons 1 and 2 are :-

$$t_{1,arr} = t_1 + d/c$$

$$t_{2,arr} = t_1 + \Delta t + \frac{d}{c} - \frac{u\cos\phi}{c}\Delta t$$

$$\Rightarrow \Delta t_{arr} = \Delta t\left[1 - \frac{u}{c}\cos\phi\right]$$

As measured by an Earthbound observer, arrival times are dilated, by a factor $\gamma \sim \frac{1}{\sqrt{1-u^2/c^2}}$ (since $\phi \simeq 0$)

Hence $$\Delta t' = \frac{\Delta t}{\sqrt{1-u^2/c^2}}\left[1 - u\cos\phi/c\right]$$

b) For $\cos\phi \simeq 1$, $$\Delta t' = \frac{\Delta t}{\sqrt{1-u^2/c^2}}\left[1 - u/c\right]$$

Writing $\beta = u/c$, $\gamma = (1-\beta^2)^{-1/2}$

$$\Rightarrow \beta^2 = \left(1 - \frac{1}{\gamma^2}\right)$$

$$\Leftrightarrow \beta = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} \simeq 1 - \frac{1}{2\gamma^2} \quad \text{for } \gamma \gg 1$$

Thus $$\Delta t' \simeq \frac{\Delta t\left[1 - \left(1 - \frac{1}{2\gamma^2}\right)\right]}{\sqrt{1-\left(1-\frac{1}{\gamma^2}\right)}} = \Delta t \cdot \frac{1}{2\gamma^2} \cdot \gamma$$

$$= \frac{\Delta t}{2\gamma} \quad \text{as required}$$

c) Blazars show rapid continuum variability in <u>our</u> frame; this is consistent with their rest frame variability being over much longer timescales, but "speeded up" in our frame by the high Lorentz factor of their approaching jet.

d) Let $\Delta t'$ = time between arrival of successive wavefronts, as measured in Earth's frame

$$\Delta t' = \frac{\lambda'}{c} = \frac{1}{\nu'} \simeq \frac{\Delta t}{2\gamma} = \frac{1}{2\gamma\nu}$$

i.e. $\nu' = 2\gamma\nu$ as required

e) Luminosity is increased by two factors :-
1) beaming, squeezing emission into a narrow cone
2) time dilation, shortening $\Delta t'$ in Earth frame

f_1 => isotropic emission over 2π steradians squeezed into a solid angle $\pi\theta^2 = \frac{\pi}{\gamma^2}$

=> luminosity increased by $f_1 \sim 2\gamma^2$

f_2 increases luminosity by an additional $2\gamma^2$
(since $L_{obs} = \Delta E / \Delta t'$)

$\therefore \quad f = (2\gamma)^2$

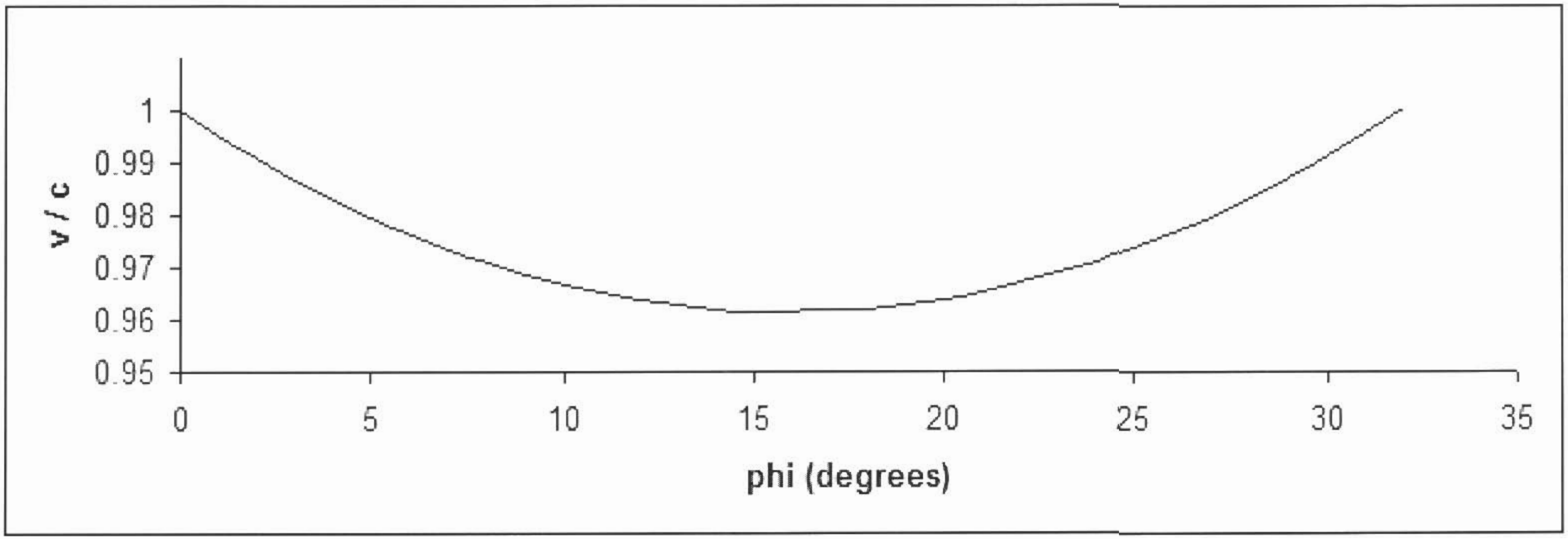
f) $$\frac{L_{approaching}}{L_{receding}} \sim (2\gamma)^4$$

Suppose $u_{app} = 25c$ => $\beta_{app} = 25$

$$\therefore \quad \beta_{min} \geq \sqrt{\frac{25^2}{1+25^2}} = 0.9992$$

$$\gamma \geqslant 25$$

$$\Rightarrow \frac{L_{app}}{L_{rec}} \gtrsim (50)^4 \sim 6 \times 10^6$$



Plot of $\beta = v/c$ versus phi, for question 1.(f)