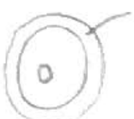
1)



dr Mass of shell, $dm = 4\pi r^2 \rho\, dr$

P.E. of shell from mass M_r, interior to r

$$du = -\frac{G M_r 4\pi r^2 \rho\, dr}{r} = -4\pi G M_r \rho\, r\, dr$$

$$\Rightarrow \quad U = \int_0^R du = -4\pi G \int_0^R M_r \rho\, r\, dr$$

$$M_r = \frac{4}{3}\pi r^3 \rho \quad \Rightarrow \quad U = -4\pi G \int_0^R \frac{4}{3}\pi \rho^2 r^4\, dr$$

$$= \frac{-16\pi^2 G \rho^2}{3}\left[\frac{r^5}{5}\right]_0^R = -\frac{16}{15}\pi^2 G \rho^2 R^5$$

Using $M = \frac{4}{3}\pi R^3 \rho \Rightarrow \rho = \frac{3M}{4\pi R^3}$

$$U = \frac{-16\pi^2}{15} G \frac{9M^2}{16\pi^2 R^6} R^5 = -\frac{9GM^2}{15R} = -\frac{3}{5}\frac{GM^2}{R}$$

$\langle K \rangle \simeq \frac{1}{2} N m \langle v^2 \rangle$ assuming all N stars have the same m

Take $Nm = M$

$$2\langle K \rangle = M\langle v^2 \rangle = \frac{3}{5}\frac{GM^2}{R}$$

Assuming the velocities of the cluster stars are isotropic,

$$\sigma_0^2 \simeq \frac{1}{3}\langle v^2 \rangle$$

$$\simeq \frac{GM}{5R} \quad \text{as required}$$

2) $$L = \int_0^\infty 2\pi I(R_e)\, e^{-7.67\left[\left(\frac{R}{R_e}\right)^{1/4} - 1\right]} R\, dR$$

Define $z = R/R_e$

$$\Rightarrow \quad L = R_e^2 \int_0^\infty 2\pi I(R_e)\, e^{-7.67\left[z^{1/4}\right]}\, e^{7.67}\, z\, dz$$

Put $t = 7.67 z^{1/4} \Rightarrow z = \dfrac{t^4}{(7.67)^4} \Rightarrow dz = \dfrac{4t^3\, dt}{(7.67)^4}$

$$\therefore \quad L = \frac{8\pi}{(7.67)^8} I(R_e) R_e^2 e^{7.67} \int_0^\infty t^7 e^{-t}\, dt$$

$$\int_0^\infty t^n e^{-t}\, dt = n!$$ (you would be given this result in an exam)

$$\Rightarrow \quad L = \frac{8!\, e^{7.67}}{(7.67)^8} \pi R_e^2 I(R_e) \simeq 7.22\, \pi R_e^2 I(R_e)$$

The Light within R_e satisfies

$$L(R_e) = \int_0^{7.67} \frac{8\pi}{(7.67)^8} I(R_e) R_e^2 e^{7.67}\, t^7 e^{-7}\, dt$$

$$= \frac{8\pi R_e^2}{(7.67)^8} I(R_e)\, e^{7.67} I_7$$

Thus $\dfrac{L(R_e)}{L_{tot}} = \dfrac{I_7}{7!}$ as required

$$I_n = \int_0^{7.67} t^n e^{-t}\,dt$$

$$= \left[-t^n e^{-t}\right]_0^{7.67} + n\int_0^{7.67} t^{n-1} e^{-t}\,dt$$

$$= -(7.67)^n e^{-7.67} + n I_n$$

$$I_0 = \int_0^{7.67} e^{-t}\,dt = \left[-e^{-t}\right]_0^{7.67} = 1 - e^{-7.67}$$

$$I_1 = -7.67\,e^{-7.67} + 1 - e^{-7.67}$$

$$I_2 = -(7.67)^2 e^{-7.67} + 2\left[1 - e^{-7.67} - 7.67\,e^{-7.67}\right]$$

$$I_3 = -(7.67)^3 e^{-7.67} + 3\left[2\left(1 - e^{-7.67} - 7.67e^{-7.67}\right) - (7.67)^2 e^{-7.67}\right]$$

$$\vdots$$

$$I_7 = -(7.67)^7 e^{-7.67} + 7\Big(6\Big(5\Big(4\Big(3\Big(2\left(1 - e^{-7.67} - 7.67e^{-7.67}\right) - 7.67^2 e^{-7.67}\Big) - 7.67^3 e^{-7.67}\Big) - 7.67^4 e^{-7.67}\Big) - 7.67^5 e^{-7.67}\Big) - 7.67^6 e^{-7.67}\Big)$$

$$= 2520$$

$$\Rightarrow \frac{L(R_e)}{L_{tot}} = 0.5$$

$$M_r = 4\pi\int_0^r r^2 \rho(r)\,dr = 4\pi\int_0^r \frac{r^2}{r^2+a^2}\,dr\,\frac{V_H^2}{4\pi G}$$

$$= \frac{V_H^2}{G}\int_0^r \frac{r^2\,dr}{r^2+a^2}$$

$$= \frac{V_H^2}{G}\int_0^r dr - \frac{V_H^2}{G}a^2\int \frac{dr}{a^2+r^2}$$

$$= \frac{V_H^2 r}{G} - \frac{V_H^2 a}{G}\tan^{-1}\left(\frac{r}{a}\right) = \frac{V_H^2}{G}\left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]$$

$$V^2(r) = \frac{GM_r}{r}$$

$$= V_H^2 \left[1 - \left(\frac{a}{r}\right) \tan^{-1}\left(\frac{r}{a}\right) \right]$$

4) $M(<R) \simeq \frac{5 R \sigma_0^2}{G}$ (assuming a spherical distribution)

$$R = \frac{0.2 \times \pi}{3600 \times 180} \times 825 \times 3.09 \times 10^{19} = 2.47 \times 10^{16} \text{ m}$$

$$M(<R) = 1.07 \times 10^{38} \text{ kg}$$

$$= 5.36 \times 10^{7} M_{\odot}$$

5) $R = \frac{0.2 \times \pi}{3600 \times 180} \times 16 \times 3.09 \times 10^{22} = 4.79 \times 10^{17} \text{ m}$

For circular orbital motion we use $M(<R) \simeq \frac{V^2 R}{G}$

$$V = 5 \times 10^5 \text{ ms}^{-1}$$

$$M(<R) \simeq 1.79 \times 10^{39} \text{ kg}$$

$$\simeq 9 \times 10^{8} M_{\odot}$$

Typical atmospheric seeing is $\sim 1''$, so HST was required to resolve the core down to $0.25''$