6. Data Acquisition

In section 5 we approximated the continuous function $h(t)$ and its FT $H(f)$ by a finite set of $N+1$ discretely sampled values.

How good is this approximation? The answer depends on the form of $h(t)$ and $H(f)$. In this short section we will consider:

1. under what conditions we can reconstruct $h(t)$ and $H(f)$ ***exactly*** from a set of discretely sampled points?

2. what is the minimum **sampling rate** (or density, if h is a spatially varying function) required to achieve this exact reconstruction?

3. what is the effect on our reconstructed $h(t)$ and $H(f)$ if our data acquisition does *not* achieve this minimum sampling rate?

6.1 The Nyquist – Shannon Sampling Theorem

Suppose the function $h(t)$ is **bandwidth limited**. This means that the FT of $h(t)$ is non-zero over a finite range of frequencies.

i.e. there exists a **critical frequency** f_c such that

$$H(f) = 0 \quad \text{for all} \quad |f| \geq f_c \qquad (6.1)$$

The **Nyquist – Shannon Sampling Theorem** (NSST) is a very important result from information theory. It concerns the representation of $h(t)$ by a set of discretely sampled values

$$h_k \equiv h(t_k) \quad \text{where} \quad t_k = k\Delta, \qquad k = \ldots,-2,-1,0,1,2,\ldots \qquad (6.2)$$

The NSST states that, provided the sampling interval Δ satisfies

$$\boxed{\Delta = 1/2 f_c} \quad \text{or less} \qquad (6.3)$$

then we can **exactly** reconstruct the function $h(t)$ from the discrete samples $\{h_k\}$. It can be shown that

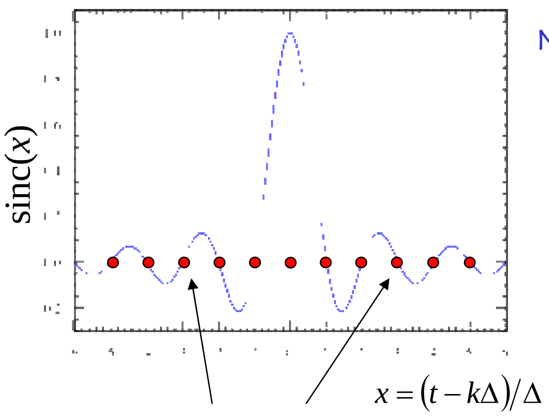
$$\boxed{h(t) = \Delta \sum_{k=-\infty}^{+\infty} h_k \frac{\sin[2\pi f_c (t - k\Delta)]}{\pi(t - k\Delta)}} \qquad (6.4)$$

f_c is also known as the **Nyquist frequency** and $\Delta^{-1} = 2 f_c$ is known as the **Nyquist rate**.

We can re-write equation (6.4) as

$$h(t) = \sum_{k=-\infty}^{+\infty} h_k \frac{\sin[\pi(t - k\Delta)/\Delta]}{\pi[(t - k\Delta)/\Delta]} \qquad (6.5)$$

So the function $h(t)$ is the sum of the sampled values $\{h_k\}$, weighted by the ***normalised* sinc function**, scaled so that its zeroes lie at those sampled values.



Sampled values

Normalised sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \qquad (6.6)$$

(compare with Section 5)

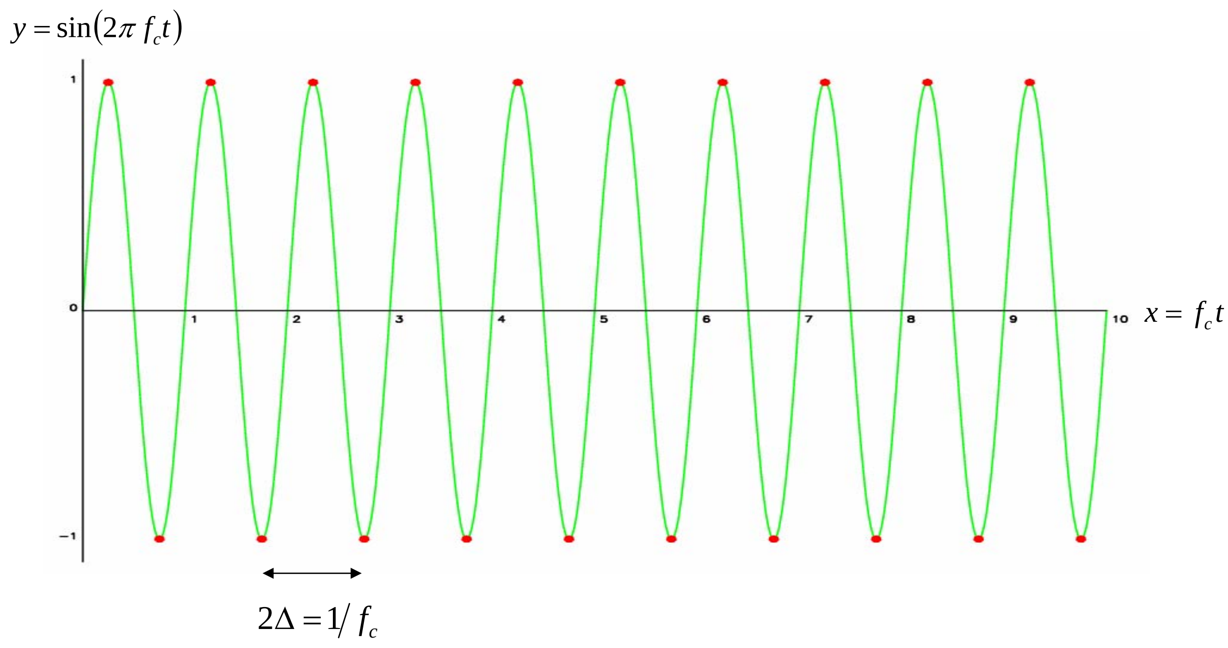
Note that when $t = k\Delta$ then $h(t) = h_k$ since $\mathrm{sinc}(0) = 1$

The NSST is a very powerful result.

We can think of the interpolating sinc functions, centred on each sampled point, as 'filling in the gaps' in our data. The remarkable fact is that they do this job *perfectly*, provided $h(t)$ is bandwidth limited. i.e. the discrete sampling incurs no loss of information about $h(t)$ and $H(f)$.

Suppose, for example, that $h(t) = \sin(2\pi f_c t)$. Then we need only sample $h(t)$ *twice* every period in order to be able to reconstruct the entire function exactly.

(Note that formally we do need to sample an *infinite number* of discretely spaced values, $\{h_k\}$. If we only sample the $\{h_k\}$ over a finite time interval, then our interpolated $h(t)$ will be approximate).



Sampling $h(t)$ at (infinitely many of) the **red** points is sufficient to reconstruct the function for all values of t, with no loss of information.

6.2 Aliasing

There is a downside, however.

If $h(t)$ is *not* bandwidth limited (or, equivalently, if we don't sample frequently enough – i.e. if the sampling rate $\Delta^{-1} < 2 f_c$) then our reconstruction of $h(t)$ and $H(f)$ is badly affected by **aliasing**.

This means that all of the power spectral density which lies *outside* the range $-f_c < f < f_c$ is spuriously moved *inside* that range, so that the FT $H(f)$ of $h(t)$ will be computed **incorrectly** from the discretely sampled data.

Any frequency component outside the range $(-f_c, f_c)$ is falsely translated (**aliased**) into that range.

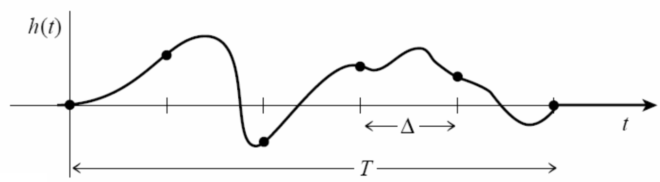
Consider $h(t)$ as shown.

Suppose $h(t)$ is zero outside the range T.

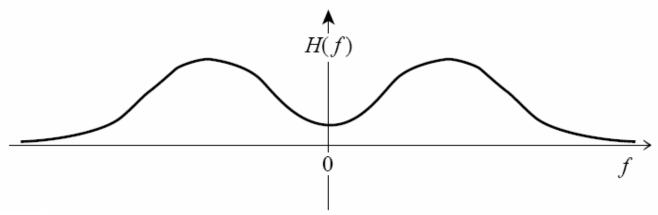
This means that $H(f)$ extends to $\pm\infty$.

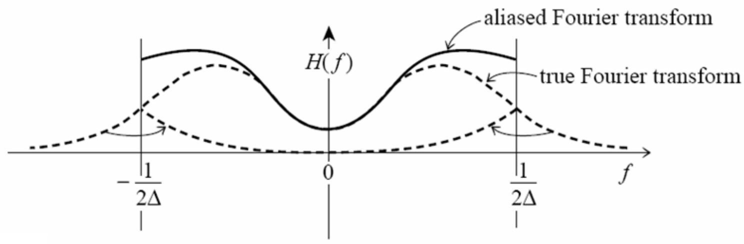
The contribution to the true FT from outside the range $(-1/2\Delta, 1/2\Delta)$ gets aliased into this range, appearing as a 'mirror image'.

Thus, at $f = \pm 1/2\Delta$ our computed value of $H(f)$ is equal to *twice* the true value.



$h(t)$ sampled at regular intervals Δ





From Numerical Recipes, Chapter 12.1

How do we combat aliasing?

- Enforce some chosen f_c e.g. by *filtering* $h(t)$ to remove the high frequency components $|f| > f_c$. *(Also known as anti-aliasing)*
- Sample $h(t)$ at a high enough rate Δ^{-1} so that $\Delta^{-1} \geq 2 f_c$ - i.e. at least two samples per cycle of the highest frequency present

To check for / eliminate aliasing *without* pre-filtering:

- Given a sampling interval Δ, compute $f_{\lim} = 1/2\Delta$
- Check if discrete FT of $h(t)$ is approaching **zero** as $f \to f_{\lim}$
- If *not*, then frequencies outside the range $(-1/2\Delta, 1/2\Delta)$ are probably being folded back into this range.
- Try increasing the sampling rate, and repeat…

6.2 Analog to Digital Conversion and Data Compression

The NSST is important in digital signal processing, in particular when taking an **analog input** and converting it into a **digital signal**. This is done using an *analog-to-digital converter* (ADC): an electronic circuit that translates continuous input signals into discrete digital output.

According to the NSST, if the analog input is bandwidth limited, then provided we sample it at the **Nyquist rate** (or higher), then the digital signal this produces has exactly the same information content as the original analog signal.

This means that, if we convert the digital signal *back* into an analog signal (using a *digital-to-analog converter*, or DAC) then we recover the original analog input signal exactly.

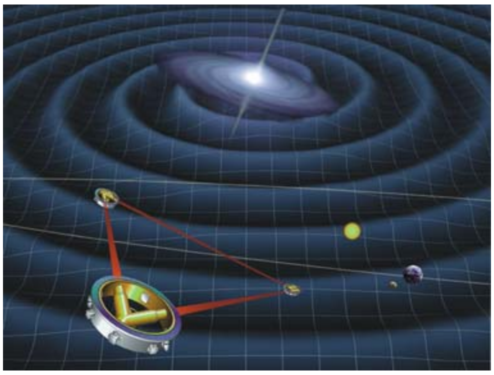
ADC is also a powerful technique for **data compression**.

Again, provided the analog input signal is bandwidth limited, by converting it to a digital signal, sampled at the Nyquist rate or better, we can compress the information content of the original analog input into the minimum number of bits of information, with no loss.

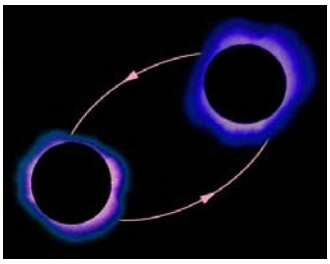
This can be particularly important for spacecraft, where we want to transmit the acquired astronomical data as cheaply and efficiently as possible, without losing valuable information content.

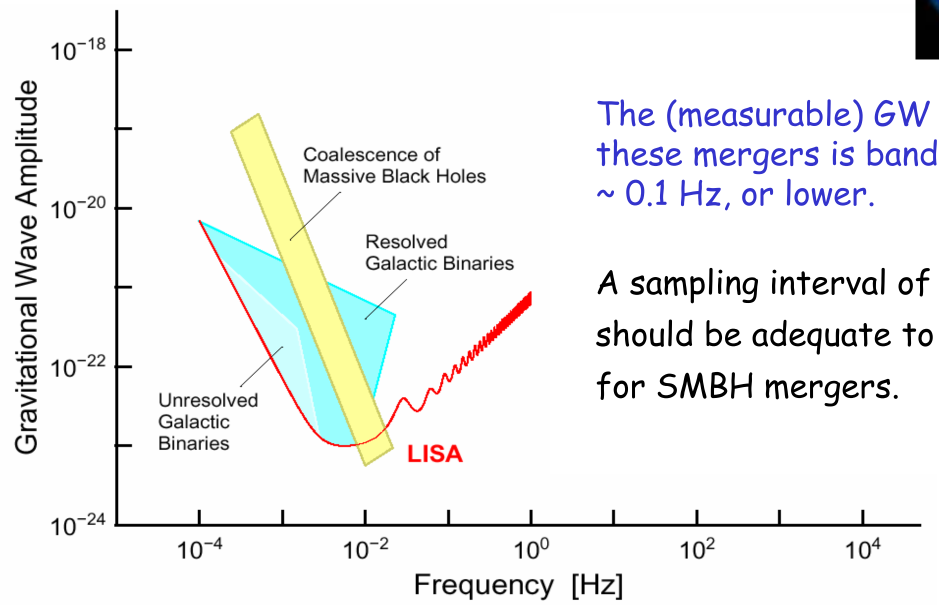
Consider, for example, the proposed LISA satellite

NASA / ESA, planned for ~2015 or later, to measure gravitational waves from space.



A major target of LISA will be to detect mergers of supermassive black holes in the cores of distant galaxies.





The (measurable) GW signal from these mergers is bandwidth limited to ~ 0.1 Hz, or lower.

A sampling interval of $\Delta \sim 1/0.2 = 5\,\mathrm{sec}$ should be adequate to describe the $h(t)$ for SMBH mergers.

In fact, **LISA** will be sampled every **~3.76** seconds, implying a Nyquist frequency of **0.133 Hz**.