Astronomy A345H

Astronomical Data Analysis I: Supplementary Handout

Proof of Poisson variable probability density function (Non-examinable)

The three postulates which define a Poisson RV with pdf $p(r,t) \;=\; \frac{(\mu t)^r}{r!}\, e^{-\mu t}$ are:

1. The probability of an event occuring in time interval, t, is independent of the past history of events prior to t
2. For small interval, δt, there is an intrinsic rate, μ (> 0) such that the probability of a single event in δt, $p(1, \delta t) = \mu\delta t + o(\delta t)$
3. The probability of two or more events happening at the same time is zero, i.e. $p(r, \delta t) = o(\delta t)$, for all $r \geq 2$.

(Here $o(\delta t)$ represents any function such that $o(\delta t)/\delta t \to 0$ as $\delta t \to 0$)

From postulate (1)

$$p(0, t + \delta t) \;=\; p(0,t)p(0,\delta t)$$

We can also write $p(0, \delta t)$ as

$$\begin{aligned} p(0,\delta t) &= 1 - \sum_{i=1}^{\infty} p(i, \delta t) \\ &= 1 - \mu\delta t - o(\delta t) \end{aligned}$$

(which follows from postulate (3) since $p(i, \delta t) = o(\delta t)$ for $i \geq 2$)

Hence, we may write

$$\begin{aligned} p(0, t+\delta t) &= p(0,t)\,[1 - \mu\delta t - o(\delta t)] \\ \Leftrightarrow \quad \frac{p(0,t+\delta t) - p(0,\delta t)}{\delta t} &= -\mu\, p(0,t) \;-\; \frac{o(\delta t)}{\delta t}\, p(0,t) \\ \Leftrightarrow \quad \frac{d\,p(0,t)}{dt} &= -\mu\, p(0,t) \end{aligned}$$

in the limit as $\delta t \to 0$. Solving this differential equation we obtain

$$p(0,t) = Ae^{-\mu t}$$

for some constant A. Since $p(0,0) = 1$, it follows that $A = 1$.

Consider now $p(r, t + \delta t)$ where $r \geq 1$. Postulates (1) and (3) imply that (in the limit as $\delta t \to 0$)

$$\begin{aligned} p(r, t+\delta t) &= p(r,t)\,p(0,\delta t) \;+\; p(r-1,t)\,p(1,\delta t) \;+\; o(\delta t) \\ \Leftrightarrow \quad \frac{p(r,t+\delta t) - p(r,\delta t)}{\delta t} &= -\mu\, p(r,t) \;+\; \mu\, p(r-1,t) \\ \Leftrightarrow \quad \frac{d\,p(r,t)}{dt} &= -\mu\, p(r,t) \;+\; \mu\, p(r-1,t) \end{aligned}$$

We show by induction that $p(r,t) \;=\; \frac{(\mu t)^r}{r!}\, e^{-\mu t}$ is a solution to this equation, for all r. Consider $r = 1$.

$$
\begin{aligned}
\frac{d\,p(1,t)}{dt} &= \mu\, e^{-\mu t} - \mu\, t e^{-\mu t} \\
&= \mu\, p(0,t) - \mu\, p(1,t)
\end{aligned}
$$

Hence, the assumed functional form of $p(r,t)$ is a solution to the above equation for $r = 1$. Suppose now that $p(s,t) = \frac{(\mu t)^s}{s!}\, e^{-\mu t}$ is a solution to the equation, for some $s \geq 1$. Consider now the case where $r = s + 1$.

$$
\begin{aligned}
\frac{d\,p(s+1,t)}{dt} &= \frac{(s+1)\mu^{s+1} t^s e^{-\mu t}}{(s+1)!} - \frac{\mu^{s+2} t^{s+1} e^{-\mu t}}{(s+1)!} \\
&= \mu\, p(s,t) - \mu\, p(s+1,t)
\end{aligned}
$$

Hence $p(r,t)$ is also a solution for $r = s + 1$. $p(r,t)$ thus satisfies the Poisson postulates for all r, and the proof is complete.