Astronomical Data Analysis I   Examples 4

1) $I_1 = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx \quad = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$

Suppose $m = n$.

Then $I_1 = \frac{1}{2}\left[\int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} \cos 2mx \, dx\right]$

$= \pi - \frac{1}{2}\left[\frac{1}{2m} \sin 2mx\right]_{-\pi}^{\pi}$

$= \pi$

Suppose $m \neq n$

Then $I_1 = \frac{1}{2}\left[\frac{1}{m-n}\left[\sin 2(m-n)x\right]_{-\pi}^{\pi} - \frac{1}{m+n}\left[\sin 2(m+n)x\right]_{-\pi}^{\pi}\right]$

$= 0$

Hence $I_1 = \pi \delta_{mn}$ as required

Cosine results follow using, $\cos mx \cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$

$I_3 = \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(m+n)x + \sin(m-n)x\right] dx$

Suppose $m = n \Rightarrow I_3 = \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} + 0 = 0$

Suppose $m \neq n \Rightarrow I_3 = \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0$

2) (a) $H(-f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i f t}\, dt$

$$= \int_{-\infty}^{\infty} h(t) \cos 2\pi f t\, dt - i \int_{-\infty}^{\infty} h(t) \sin 2\pi f t\, dt \qquad \text{(since } h(t) \text{ is real)}$$

$$= [H(f)]^*$$

(b) $h(t) \equiv i\, h_I(t)$

$$H(-f) = \int h(t) \cos 2\pi f t\, dt - i \int h(t) \sin 2\pi f t\, dt$$

$$= i \int h_I(t) \cos 2\pi f t\, dt + \int h_I(t) \sin 2\pi f t\, dt$$

$$H(f)^* = \left[\int i\, h_I(t) \cos 2\pi f t\, dt - \int h_I(t) \sin 2\pi f t\, dt \right]^*$$

$$= - \left[i \int h(t) \cos 2\pi f t\, dt + \int h_I(t) \sin 2\pi f t\, dt \right]$$

Hence, $H(-f) = -[H(f)]^*$

(c) $h(t)$ even $\Longleftrightarrow$ $h(t) = h(-t)$

$$H(-f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i f t}\, dt$$

$$= \int_{-\infty}^{\infty} h(-y)\, e^{2\pi i f y}\, dy \qquad \text{putting } y = -t$$

$$= H(f) \qquad \text{(since } h(t) \text{ even)} \qquad \text{so } H(f) \text{ is also even}$$

(d) $H(-f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i f t}\, dt$

$$= \int_{-\infty}^{\infty} h(-y)\, e^{2\pi i f y}\, dy$$

$$= -H(f) \qquad \text{(since } h(t) \text{ odd)} \qquad \text{so } H(f) \text{ is also odd}$$

(e) From (a) $H(-f) = [H(f)]^*$

From (c) $H(-f) = [H(f)]$ i.e $H(f)$ is even

$\Rightarrow$ $H(f)^* = H(f)$ i.e. $H(f)$ is real and even

(f) From (a) $H(-f) = [H(f)]^*$

From (d) $H(-f) = -H(f)$ i.e. $H(f)$ is odd

$\Rightarrow$ $H(f) = -H(f)^*$ i.e. $H(f)$ is imaginary and odd

(g) From (a) $H(-f) = H(f)$ i.e. $H(f)$ is even

From (b) $H(-f) = -[H(f)]^*$

$\Rightarrow$ $H(f) = -H(f)^*$ i.e. $H(f)$ is imaginary and even

(h) From (d) $H(-f) = -H(f)$ i e $H(f)$ is odd

From (b) $H(-f) = -H(f)^*$

$\Rightarrow$ $H(f) = H(f)^*$ i.e. $H(f)$ is real and odd.

3) $$\widetilde{h(at)} = \int_{-\infty}^{\infty} h(at)\, e^{2\pi i f t}\, dt \qquad a > 0$$

$$= \int_{-\infty}^{\infty} h(at)\, e^{2\pi i (f/a) at}\, dt$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} h(at)\, e^{2\pi i (f/a)\, at}\, d(at) = \frac{1}{a}\int_{-\infty}^{\infty} h(y)\, e^{2\pi i (f/a) y}\, dy$$

$$= \frac{1}{a} H(f/a) = \frac{1}{|a|} H(f/a)$$

Now suppose $a < 0$

$$\widetilde{h(at)} = \frac{1}{a}\int_{\infty}^{-\infty} h(y)\, e^{2\pi i (f/a) y}\, dy$$

$$= -\frac{1}{a} H(f/a) = \frac{1}{|a|} H(f/a) \quad \text{as required}$$

4) Write $a = 1/b \Rightarrow h(t/b) = |b|\, H(bf)$

i.e. $\frac{1}{|b|} h(t/b) = H(bf)$ as required

5) $$\widetilde{h(t-t_0)} = \int_{-\infty}^{\infty} h(t-t_0)\, e^{2\pi i f t}\, dt$$

Put $y = t - t_0 \Rightarrow t = y + t_0$

$$\widetilde{h(t-t_0)} = \int_{-\infty}^{\infty} h(y)\, e^{2\pi i f y}\, e^{2\pi i f t_0}\, dy = H(f)\, e^{2\pi i f t_0}$$

6) $$\widetilde{h(t)\, e^{-2\pi i f_0 t}} = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i (f - f_0) t}\, dt = H(f - f_0)$$

7) $$\widetilde{(g*h)(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s)\, h(t-s)\, ds\, e^{2\pi i f t}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s)\, h(y)\, ds\, e^{2\pi i f (y+s)}\, dy \qquad y = t - s$$

$$= \int_{-\infty}^{\infty} g(s)\, e^{2\pi i f s}\, ds \int_{-\infty}^{\infty} h(y)\, e^{2\pi i f y}\, dy$$

$$= \widetilde{g(t)}\ \widetilde{h(t)} \quad \text{as required}$$

8) $$\widetilde{Corr(g,h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s+t)\,h(s)\,ds\; e^{2\pi i f t}\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)\,h(s)\,ds\; e^{2\pi i f y}\, e^{-2\pi i f s}\,dy \qquad y = s+t$$

$$= \int_{-\infty}^{\infty} g(y)\, e^{2\pi i f y}\,dy \int_{-\infty}^{\infty} h(s)\, e^{-2\pi i f s}\,ds$$

$$= \widetilde{g(t)}\left[\widetilde{h(t)}\right]^{*} \qquad (\text{for real } h(t))$$