1) $p(x) = x^2/9 \qquad 0 < x < 3$

We want to find $p(y)$ s.t. $p(y)\,dy = p(x)\,dx$

i.e. $p(y) = p(x) \Big/ \left| \frac{dy}{dx} \right|$

$$p(y) = \frac{x^2}{9} \Big/ 3x^2 = \frac{1}{27} \qquad 0 < y < 27,$$

zero elsewhere

2) $p(x) = 2x\,e^{-x^2} \qquad 0 < x < \infty$

We want to find $p(y)$ s.t. $p(y)\,dy = p(x)\,dx$

i.e. $p(y) = p(x) \Big/ \left| \frac{dy}{dx} \right|$

$$= 2x\,e^{-x^2} \Big/ 2x$$

$$= e^{-x^2} = e^{-y} \qquad 0 < y < \infty,$$

zero elsewhere

3) $p(x) = \frac{1}{\pi}, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$p(y) = p(x) \Big/ \left| \frac{dy}{dx} \right|$$

$$= \frac{1}{\pi} \times \frac{1}{\sec^2 x}$$

$$= \frac{1}{\pi} \cdot \frac{1}{1 + \tan^2 x} = \frac{1}{\pi} \frac{1}{1 + y^2} \qquad -\infty < y < \infty$$

$$E(y) = \int_{-\infty}^{\infty} y \frac{1}{\pi} \frac{1}{1+y^2}\,dy = \frac{1}{\pi} \int_0^{\infty} y \frac{dy}{1+y^2} + \frac{1}{\pi} \int_{-\infty}^{0} y \frac{dy}{1+y^2}$$

Put $z = y^2$, $dz = 2y\,dy$

$$, \quad E(y) = \frac{1}{2\pi}\int_0^{\infty} \frac{dz}{1+z} + \frac{1}{2\pi}\int_{\infty}^{0} \frac{dz}{1+z}$$

$$= 0$$

$$\mathrm{var}(y) = E(y^2) - [E(y)]^2$$

$$E(y^2) = \frac{1}{\pi}\int_{-\infty}^{\infty} y^2 \frac{1}{1+y^2}\,dy$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{y^2\,dy}{1+y^2} = \frac{2}{\pi}\int_0^{\infty} \frac{(1+y^2-1)}{1+y^2}\,dy$$

$$= \frac{2}{\pi}\int_0^{\infty} dy - \frac{2}{\pi}\int_0^{\infty}\frac{dy}{1+y^2}$$

$$= \frac{2}{\pi}\Big[y\Big]_0^{\infty} - \frac{2}{\pi}\Big[\tan^{-1} y\Big]_0^{\infty}$$

$$= \infty$$ i.e. the Cauchy distribution has infinite variance

1 (a) $p(\alpha, \beta)$ = pdf of GRB distribution

We know that $p(\alpha,\beta) = \dfrac{d\Omega}{4\pi}$ (since $\iint p(\alpha,\beta)\,d\alpha\,d\beta = 1$)

$$p(\alpha)\,d\alpha = \int_{\beta} p(\alpha,\beta)\,d\beta\,d\alpha$$

$$= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \cos\beta\,d\beta\,d\alpha$$

$$= \frac{1}{4\pi}\Big[\sin\beta\Big]_{-\pi/2}^{\pi/2}\,d\alpha = \frac{1}{2\pi}\,d\alpha$$

so $p(\alpha) = \dfrac{1}{2\pi}$ $\quad 0 \le \alpha \le 2\pi$

$$p(\beta)\,d\beta = \int_\alpha p(\alpha,\beta)\,d\alpha\,d\beta$$

$$= \frac{1}{4\pi}\cos\beta \int_0^{2\pi} d\alpha\,d\beta$$

$$\Rightarrow \quad p(\beta) = \frac{1}{2}\cos\beta \qquad -\frac{\pi}{2} < \beta < \frac{\pi}{2}$$

(b) We want to find y s.t. $p(y) = \frac{1}{2}$

where $p(y)\,dy = p(\beta)\,d\beta$

i.e. $\frac{1}{2} = \frac{1}{2}\cos\beta \Big/ \left|\frac{dy}{d\beta}\right|$

$$\frac{dy}{d\beta} = \cos\beta$$

$$\Rightarrow \quad y = f(\beta) = \sin\beta$$

5) a) Likelihood : $L(\mu) = \dfrac{\mu^r e^{-\mu}}{r!}$

$$l = \ln L = \ln(\mu^r) - \mu - \ln r! = r\ln\mu - \mu - \ln r!$$

$$\frac{\partial l}{\partial \mu} = \frac{r}{\mu} - 1 = 0 \iff \hat{\mu}_{ML} = r$$

b) Likelihood $L(\mu) = \prod_{i=1}^{n} \dfrac{\mu^{r_i} e^{-\mu}}{r_i!}$ assuming μ constant throughout run.

$$l = \ln L = \sum_{i=1}^{n} r_i \ln\mu - n\mu - \sum \ln r_i!$$

$$\frac{\partial l}{\partial \mu} = \frac{1}{\mu}\sum_{i=1}^{n} r_i - n$$

$$= 0 \iff \hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

6) a) $p(r|\theta) = C\,\theta^r (1-\theta)^{n-r}$

$$\ell(\theta) = \ln p(r|\theta)$$

$$= \ln C + r\ln\theta + (n-r)\ln(1-\theta)$$

b) $$\frac{\partial \ell}{\partial \theta} = \frac{r}{\theta} - \frac{n-r}{1-\theta}$$

$$= 0 \iff \frac{r}{\theta} = \frac{n-r}{1-\theta}$$

$$\iff r(1-\theta) = (n-r)\theta$$

$$\iff r = n\theta$$

$$\iff \hat{\theta}_{ML} = \frac{r}{n}$$

c) $p(\theta|r) \propto p(r|\theta)\,p(\theta)$ where the constant of proportionality is independent of θ

If we take $p(\theta)$ = constant, then

$$\frac{\partial p(\theta|r)}{\partial\theta} = 0 \iff \frac{\partial p(r|\theta)}{\partial\theta} = 0.$$

But $p(r|\theta)$ is maximised $\iff$ $\ell(\theta)$ is maximised

i.e. when $\hat{\theta}_{ML} = \frac{r}{n}$ as required

d) $$p(\theta|r) = K\,\theta^r(1-\theta)^{n-r}\left[1-4(\theta-0.5)^2\right]$$

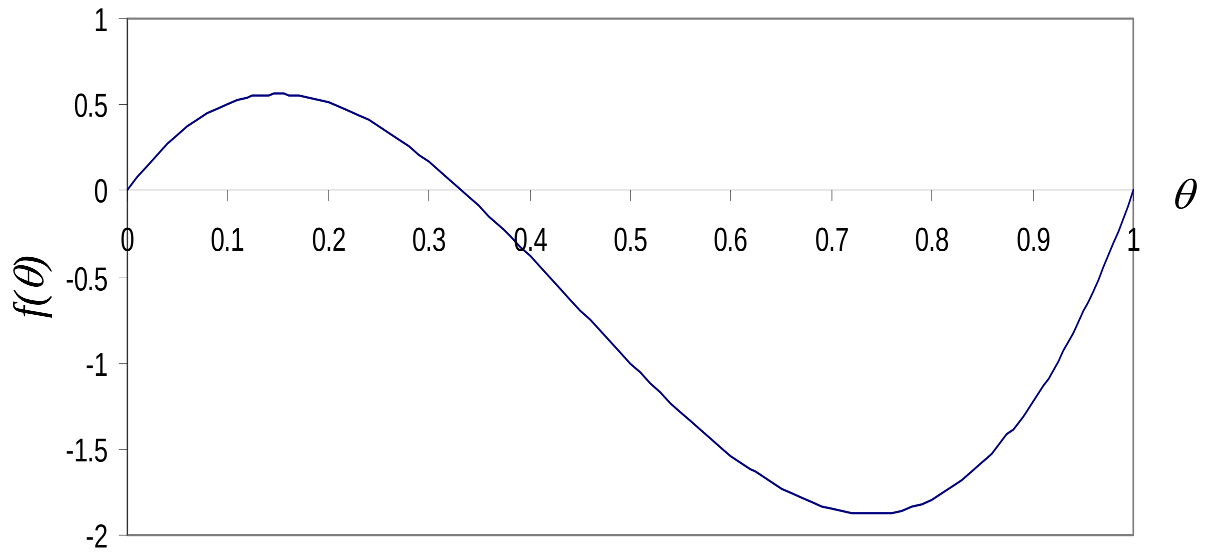
$$\ell = \ln(p(\theta|r)) = \ln K + r\ln\theta + (n-r)\ln(1-\theta) + \ln\left[1-4(\theta-0.5)^2\right]$$

$$\frac{\partial \ell}{\partial\theta} = \frac{r}{\theta} - \frac{(n-r)}{1-\theta} - \frac{8(\theta-0.5)}{\left[1-4(\theta-0.5)^2\right]}$$

$$= 0 \iff r(1-\theta)\left[1-4(\theta-0.5)^2\right] - (n-r)\theta\left[1-4(\theta-0.5)^2\right] - 8(\theta-0.5)\,\theta(1-\theta) = 0$$

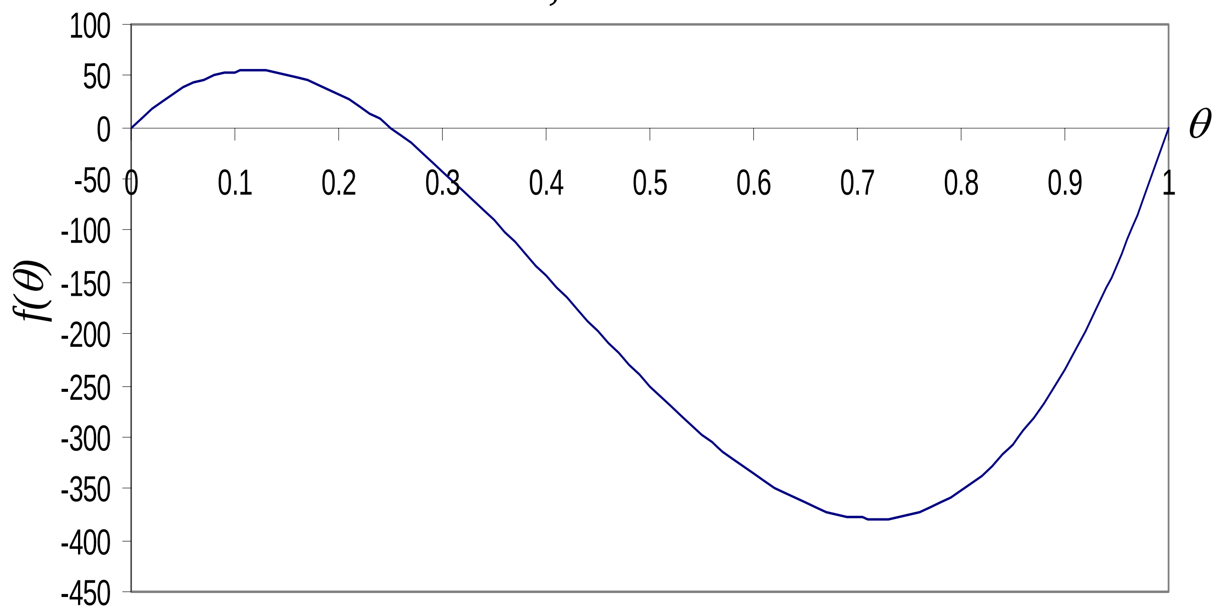
(e)

r=1, n=4



(f)

r=248, n=1000



(g) In part (e), the posterior is still dominated by the prior, which ‘pulls’ the peak in the posterior distribution towards higher values of theta.

In part (f), on the other hand, the posterior is dominated by the likelihood, so that, despite the choice of prior which strongly weights theta = 0.5, the posterior peaks very close to the maximum of the likelihood.