Astronomical Data Analysis I
Examples Sheet 1, Model Answers

Q.1

We require $\displaystyle\int_{L_*}^{\infty} p(L)\,dL = 1$

$$\int_{L_*}^{\infty} p(L)\,dL = A\int_{L_*}^{\infty} e^{-(L-L_*)/\Delta}\,dL$$

$$= A\int_{0}^{\infty} e^{-y}\,dy\,\Delta \qquad \text{putting } y=(L-L_*)/\Delta$$

$$= A\Delta\left[-e^{-y}\right]_0^{\infty} = A\Delta \qquad \Rightarrow \qquad A = \frac{1}{\Delta}$$

$$E(L) = \int_{L_*}^{\infty} L\,p(L)\,dL$$

$$= \frac{1}{\Delta}\int_{L_*}^{\infty} L\,e^{-(L-L_*)/\Delta}\,dL = \frac{1}{\Delta}\int_{0}^{\infty}(\Delta y + L_*)\,e^{-y}\,\Delta\,dy$$

$$E(L) = \int_0^{\infty}\Delta\,y\,e^{-y}\,dy + L_*\int_0^{\infty} e^{-y}\,dy$$

$$= \Delta\,\Gamma(2) + L_*\,\Gamma(1)$$

$$= \Delta + L_*$$

Similarly,

$$E(L^2) = \frac{1}{\Delta}\int_{L_*}^{\infty} L^2 e^{-(L-L_*)/\Delta}\,dL = \frac{1}{\Delta}\int_0^{\infty}(\Delta y + L_*)^2 e^{-y}\,\Delta\,dy$$

$$= \Delta^2\,\Gamma(3) + 2L_*\Delta\,\Gamma(2) + L_*^2\,\Gamma(1)$$

$$= 2\Delta^2 + 2\Delta L_* + L_*^2$$

Hence $\quad \text{var}(L) = E(L^2) - [E(L)]^2$

$$= 2\Delta^2 + 2\Delta L_* + L_*^2 - (\Delta + L_*)^2$$

$$= \Delta^2$$

$$\text{cdf of } L = P(L) = \int_{-\infty}^{L} p(L')\,dL' = \frac{1}{\Delta}\int_{L_*}^{L} e^{-(L-L_*)/\Delta}\,dL$$

$$= \int_0^{(L-L_*)/\Delta} e^{-y}\,dy = \left[-e^{-y}\right]_0^{(L-L_*)/\Delta}$$

$$= 1 - \exp\left[-\frac{(L-L_*)}{\Delta}\right]$$

Median of L is the value for which $P(L) = 0{\cdot}5$

i.e. L_{MED} satisfies $\quad 1 - \exp\left[-\frac{(L_{MED} - L_*)}{\Delta}\right] = 0{\cdot}5$

$$\Longleftrightarrow \frac{-(L_{MED} - L_*)}{\Delta} = \ln\frac{1}{2} = -\ln 2$$

$$\Longleftrightarrow L_{MED} = L_* + \Delta\ln 2$$

$$= L_* + 0{\cdot}693\,\Delta$$

$$\Rightarrow L_{MED} < L_{MEAN}, \quad \text{since} \quad E(L) = L_* + \Delta$$

Q.2 $$p(x, y) = e^{-y} \qquad 0 < x < y < \infty$$

(a) $$p(x) = \int_x^\infty e^{-y}\,dy = \left[-e^{-y}\right]_x^\infty = e^{-x}$$

$$p(y) = \int_0^y e^{-y}\,dx = \left[x e^{-y}\right]_0^y = y e^{-y}$$

(b) $$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{1}{y}$$

$$p(y|x) = \frac{p(x,y)}{p(x)} = e^{x-y}$$

(c) $p(y|x) \neq p(y) \Rightarrow X, Y$ are <u>not</u> independent

Q.3 $$p(x_1, x_2) = 12 x_1 x_2 (1 - x_2) \qquad 0 < x_1 < 1, \; 0 < x_2 < 1$$

$$p(x_1) = \int_0^1 p(x_1, x_2)\,dx_2 = \left[6 x_1 x_2^2 - 4 x_1 x_2^3\right]_0^1 = 2x_1$$

$$p(x_2) = \int_0^1 p(x_1, x_2)\,dx_1 = \left[6 x_1^2 x_2 - 6 x_1^2 x_2^2\right]_0^1 = 6x_2 - 6x_2^2$$

$p(x_1, x_2) = p(x_1)\,p(x_2) \Rightarrow X_1, X_2$ are independent

(Note slightly different notation from question sheet)

Q.4

$$I = \int p(x)\,dx = k\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$

$$\Rightarrow I = k\int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta\,d\theta}{\sqrt{\sin^2\theta\ \cos^2\theta}} = 2k\int_0^{\pi/2} d\theta = k\pi \Rightarrow k = \frac{1}{\pi}$$

b) $$E(x) = \frac{1}{\pi}\int_0^1 \frac{x\,dx}{\sqrt{x(1-x)}} = \frac{2}{\pi}\,\frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

c) $$P(t) = \text{Prob}(X < t) = \frac{1}{\pi}\int_0^t \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi}\int_0^{\sin^{-1}\sqrt{t}} d\theta \quad (\text{putting } t = \sin^2\theta)$$

$$= \frac{2}{\pi}\sin^{-1}\sqrt{t}$$

when $t = \frac{1}{4} \Rightarrow \sin^{-1}\sqrt{t} = \frac{\pi}{6} \Rightarrow P(X < 0.25) = \frac{1}{3}$

Q.5

$$\text{var}[X] = E\left[(X-\bar{x})^2\right]$$

$$= E\left[X^2 - 2\bar{x}X + \bar{x}^2\right]$$

$$= E\left[X^2\right] - 2\bar{x}E[X] + \bar{x}^2 \quad (\text{since } \bar{x} \text{ is a constant})$$

$$= E\left[X^2\right] - 2\bar{x}^2 + \bar{x}^2$$

$$= E\left[X^2\right] - \left[E(X)\right]^2 \quad \text{as given}$$

Q.6

<u>Uniform Distrib.</u>

$$p(x) = \frac{1}{b-a} \qquad a < x < b$$

$$E(x) = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$= \frac{1}{b-a} \int_a^b x\, dx$$

$$= \frac{1}{b-a} \left[\tfrac{1}{2} x^2 \right]_a^b = \frac{1}{2} \frac{b^2 - a^2}{b-a}$$

$$= \frac{1}{2}(a+b)$$

$$var(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \frac{1}{b-a} \int_a^b x^2\, dx$$

$$= \frac{1}{3} \frac{(b^3 - a^3)}{(b-a)}$$

$$\Rightarrow \quad var(x) = \frac{1}{3}\frac{(b^3 - a^3)}{(b-a)} - \frac{1}{4}(a+b)^2$$

$$= \frac{1}{12} \cdot \frac{1}{(b-a)} \cdot \left[4b^3 - 4a^3 - 3(a^2 + 2ab + b^2)(b-a) \right]$$

$$= \frac{1}{12} \left[\frac{4b^3 - 4a^3 - 3a^2 b - 6ab^2 - 3b^3 + 3a^3 + 6a^2 b + 3b^2 a}{(b-a)} \right]$$

$$= \frac{b^3 - a^3 + 3a^2 b - 3ab^2}{12\,(b-a)}$$

$$= \frac{1}{12}(b-a)^2$$

Normal Distrib.

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

$$E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] dx$$

Put $t = \dfrac{x-\mu}{\sigma} \implies x = \sigma t + \mu$, $dx = \sigma\, dt$

$$\implies E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\sigma t + \mu)\, e^{-\frac{1}{2}t^2} \sigma\, dt \qquad (3$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-\frac{1}{2}t^2} dt + \mu \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} = \mu$$

$$E(x^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\, x^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \left(\sigma^2 t^2 + 2\sigma t \mu + \mu^2\right) e^{-\frac{1}{2}t^2} \sigma\, dt$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{1}{2}t^2} dt + \frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-\frac{1}{2}t^2} dt + \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} t^2 e^{-\frac{1}{2}t^2} dt + \mu^2$$

$$= \frac{2\sigma^2\sqrt{2}}{\sqrt{2\pi}} \int_{0}^{\infty} y^{\frac{1}{2}} e^{-y} dy + \mu^2$$

(putting $y = \frac{1}{2}t^2$, $t = \sqrt{2y}$, $dy = t\,dt$)

$$= \frac{2}{\sqrt{\pi}} \sigma^2 \Gamma\left(\frac{3}{2}\right) + \mu^2 \qquad \left(\text{where } \Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy\right)$$

$$= \frac{2}{\sqrt{\pi}} \sigma^2 \frac{1}{2} \Gamma\left(\frac{1}{2}\right) + \mu^2 \qquad \left(\text{using } \Gamma(z) = (z-1)\,\Gamma(z-1)\right)$$

$$= \sigma^2 + \mu^2 \qquad \left(\text{using } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\Rightarrow \operatorname{var}(X) = E(X^2) - [E(X)]^2 = \sigma^2$$

Note, you would be given this result in an exam

Poisson Distrib.

$$E(X) = \sum_{x=0}^{\infty} x \frac{\mu^x e^{-\mu}}{x!} = 0 + \sum_{x=1}^{\infty} \frac{x \mu^x e^{-\mu}}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{\mu^x e^{-\mu}}{(x-1)!} = \mu \sum_{x=1}^{\infty} \frac{\mu^{x-1} e^{-\mu}}{(x-1)!}$$

$$= \mu \sum_{y=0}^{\infty} \frac{\mu^y e^{-\mu}}{y!} \qquad (\text{Putting } y = x-1)$$

$$= \mu \qquad \left(\text{since } \sum_{y=0}^{\infty} \frac{\mu^y e^{-y}}{y!}\right)$$

To determine var(X) we can apply a similar change of variable

$$E[X^2 - X] = \sum_{x=0}^{\infty} \frac{x(x-1)\mu^x e^{-\mu}}{x!} = 0 + \sum_{x=2}^{\infty} \frac{\mu^x e^{-\mu}}{(x-2)!}$$

$$= \mu^2 \sum_{x=2}^{\infty} \frac{\mu^{x-2} e^{-\mu}}{(x-2)!} = \mu^2 \sum_{y=0}^{\infty} \frac{\mu^y e^{-y}}{y!} \qquad (\text{Putting } y = x-2)$$

$$= \mu^2$$

Thus $E[X^2] = \mu^2 + E[X] = \mu^2 + \mu$

$$\Rightarrow \operatorname{var}[x] = E[X^2] - [E(X)]^2 = \mu^2 + \mu - \mu^2$$

$$= \mu.$$