Astronomy A2Z

Observational Astrophysics: The Poisson Distribution

(Non-examinable proof of the mathematical form of the Poisson distribution)

Recall that we use the Poisson distribution $p(N, \tau)$ to describe the probability of receiving N photons in time τ, and

$$p(N, \tau) \;=\; \frac{(Rt)^N}{N!}\, e^{-R\tau}$$

where R is the **mean arrival rate** of photons. There are three properties which define the Poission distribution:-

1. The probability of a photon arriving in time interval, t, is independent of the past history of events prior to t – i.e. photons arrive independently in time
2. For small time interval, δt, there is an intrinsic rate, R (> 0) such that the probability of a single photon arriving in δt, $p(1, \delta t) = R\delta t$
3. The probability of two or more events happening at the same time is zero – i.e. $p(M, \delta t) \simeq 0$, for all $M \geq 2$.

From property (1), we can multiply probabilities:-

$$p(0, t + \delta t) \;=\; p(0, t)p(0, \delta t)$$

We can also write $p(0, \delta t)$

$$\begin{aligned} p(0, \delta t) &= 1 - \sum_{i=1}^{\infty} p(i, \delta t) \\ &= 1 - R\delta t \end{aligned}$$

(which follows from postulate (3) since $p(i, \delta t) \simeq 0$ for $i \geq 2$)

Hence, we can write

$$\begin{aligned} p(0, t + \delta t) &= p(0, t)\,[1 - R\delta t] \\ \Leftrightarrow \quad \frac{p(0, t + \delta t) - p(0, \delta t)}{\delta t} &= -R\,p(0, t) \qquad (1) \\ \Leftrightarrow \quad \frac{d\,p(0, t)}{dt} &= -R\,p(0, t) \end{aligned}$$

in the limit as $\delta t \to 0$. Solving this differential equation we obtain

$$p(0, t) = Ae^{-Rt}$$

for some constant A. Since $p(0, 0) = 1$, it follows that $A = 1$.

Consider now $p(N, t+\delta t)$ where $N \geq 1$. Postulates (1) and (3) imply that (in the limit as $\delta t \to 0$)

$$
\begin{aligned}
p(N, t+\delta t) &= p(N, t)\, p(0, \delta t) + p(N-1, t)\, p(1, \delta t) \\
\Leftrightarrow \quad \frac{p(N, t+\delta t) - p(N, \delta t)}{\delta t} &= -R\, p(N, t) + R\, p(N-1, t) \\
\Leftrightarrow \quad \frac{d\, p(N, t)}{dt} &= -R\, p(N, t) + R\, p(N-1, t)
\end{aligned}
$$

We can show by the mathematical technique known as induction that $p(N, t) = \frac{(Rt)^N}{N!} e^{-Nt}$ is a solution to this equation, for all N. Consider $N = 1$.

$$
\begin{aligned}
\frac{d\, p(1, t)}{dt} &= R e^{-Rt} - R t e^{-Rt} \\
&= R\, p(0, t) - R\, p(1, t)
\end{aligned}
$$

Hence, the assumed functional form of $p(N, t)$ is a solution to the above equation for $N = 1$. Suppose now that $p(s, t) = \frac{(Rt)^s}{s!} e^{-Rt}$ is a solution to the equation, for some $s \geq 1$. Consider now the case where $N = s + 1$.

$$
\begin{aligned}
\frac{d\, p(s+1, t)}{dt} &= \frac{(s+1) R^{s+1} t^s e^{-Rt}}{(s+1)!} - \frac{R^{s+2} t^{s+1} e^{-Rt}}{(s+1)!} \\
&= R\, p(s, t) - R\, p(s+1, t)
\end{aligned}
$$

Hence $p(N, t)$ is also a solution for $N = s + 1$. $p(N, t)$ thus satisfies the Poisson postulates for all N, and the proof is complete.