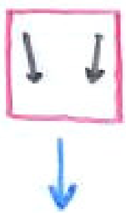
How Local is an LIF?

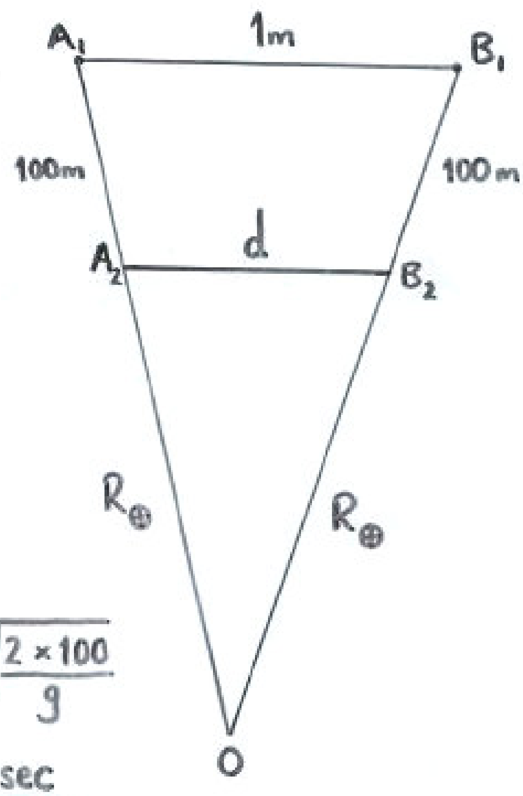
Consider again our lift in free fall

e.g. Lift begins to fall from a height of 100m



Test particles in lift fall freely towards centre of Earth

Initial separation of 1m



Take $g = 9{\cdot}81 \text{ ms}^{-2}$

Time taken to reach ground $= \sqrt{\dfrac{2 \times 100}{g}}$

$= 4{\cdot}52 \text{ sec}$

From similar triangles :- $\dfrac{R_\oplus + 100}{R_\oplus} = \dfrac{1}{d} = 1 + \dfrac{100}{R_\oplus}$

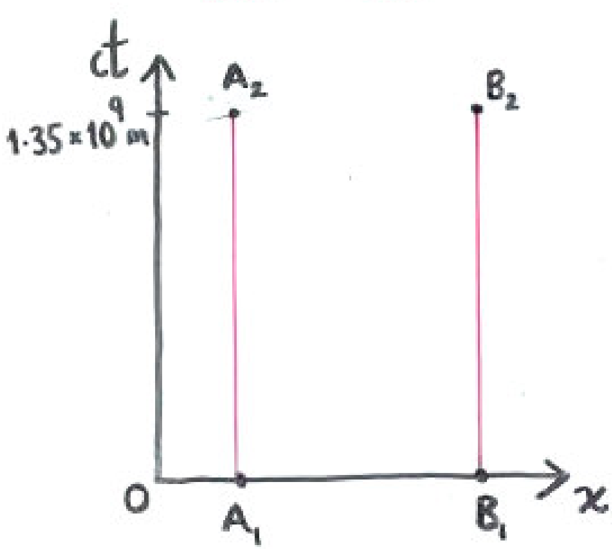
Taking $R_{\oplus} = 6 \cdot 378 \times 10^{6}$ m

$$\Rightarrow \quad d = 1 - 1 \cdot 6 \times 10^{-5}$$

i.e. Test particles move together by $1 \cdot 6 \times 10^{-5}$ m or 0·016 mm

Provided this is smaller than the accuracy of our measuring equipment, we can treat the lift as a LIF throughout the 4·52 sec of its free fall.

i.e. on a Minkowski diagram the World Lines of A and B would be approximately parallel in the LIF



In the LIF

$$A_1 B_1 = 1\text{m}$$

$$A_2 B_2 = 0 \cdot 999984\,\text{m}$$

These worldlines are **GEODESICS**

A **GEODESIC** is the trajectory followed by a test particle moving in spacetime when no non-gravitational forces act upon it

In SR geodesics are STRAIGHT LINES

i.e. in an inertial frame, trajectory satisfies

$$\frac{d^2t}{d\tau^2} = 0 \qquad \frac{d^2x}{d\tau^2} = 0 \qquad \text{etc}$$

We say that the Geodesic acceleration is zero

In GR, the Geodesic acceleration is in general non-zero, i.e. particles follow CURVED geodesics, due to the CURVATURE of spacetime

In our falling lift, LIF approximation is good because :-

Geodesic acceleration is very small

$\Leftrightarrow$ Spacetime curvature is very small

$\Leftrightarrow$ Gravitational field is very weak

Close to a more massive, more compact body (eg Sun, white dwarf, neutron star etc) spacetime curvature progressively greater

$\Downarrow$

Test particles continue to follow geodesics, but these deviate increasingly from straight lines

REMEMBER

"Spacetime tells matter how to move and matter tells spacetime how to curve"

BUT Strong EP $\Rightarrow$ we can always find a coord system in which geodesics LOCALLY are straight lines

$\Rightarrow$ Locally we can "transform away" gravity

As the curvature of spacetime increases, the region over which it looks locally flat decreases.

Underlying mathematical description of curvature is the METRIC which measures the separation of events in spacetime

SR

$$\boxed{ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)} \quad (1)$$

This is the metric of Minkowski 'flat' spacetime

Coeffs of dt^2, dx^2 etc are CONSTANT

In the curved spacetime of GR, coeffs may depend on (x, y, z, t) and there may be cross terms (e.g. $dxdt$ etc)

In A3/A4 we consider precise mathematical relation between spacetime curvature and the matter (and energy) in spacetime.

In this introductory course we content ourselves with :-

(1) How do we measure the curvature of a space ?

(2) What are the geodesics for some simple cases ?

(3) What about black holes ?!!.....

Measuring Intrinsic Curvature

How do we measure the curvature of a space intrinsically, by measurements made in the surface itself? .....

e.g. 2-d spaces.

We can "see" the difference between the sphere, plane and saddle surfaces when we view them from our 3-d space. How could we measure their curvature if we were 2-d creatures living on their surface?

Consider the ants in our diagram.

Each space is locally flat. If two ants start off on neighbouring parallel geodesics, their separation

$$\left\{\begin{array}{l}\text{remains the same}\\ \text{decreases}\\ \text{increases}\end{array}\right. \text{ for } \left\{\begin{array}{l}\text{zero}\\ \text{positive}\\ \text{negative}\end{array}\right. \text{curvature}$$

Separation of neighbouring geodesics changes

$\Updownarrow$

geodesic acceleration

$\Updownarrow$

spacetime curvature

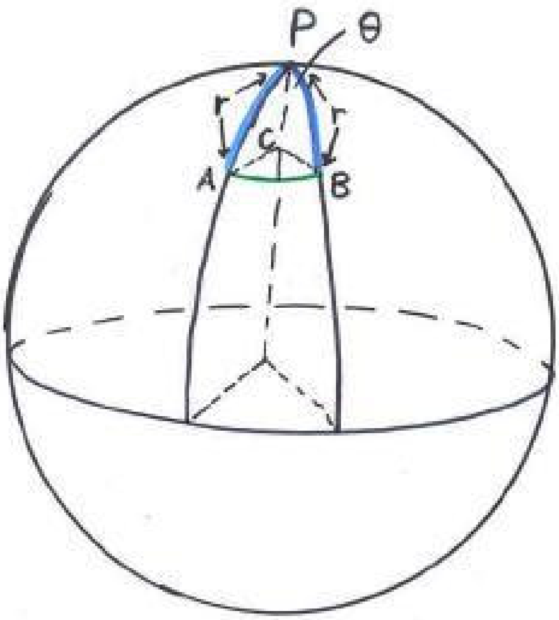
We need new maths (tensors) to explore this link any further (A3/A4)

A related approach is to consider a LOCAL measure of curvature on our 2-d surfaces: the circumference of a circle, a distance r along geodesics from a given point

In flat (Euclidean) space	$C = 2\pi r$
On a sphere (positive curvature)	$C < 2\pi r$
On a saddle (negative curvature)	$C > 2\pi r$

In formal maths, the curvature, K, is given by:-

$$K = \frac{3}{\pi} \lim_{r \to 0} \frac{2\pi r - C}{r^3}$$



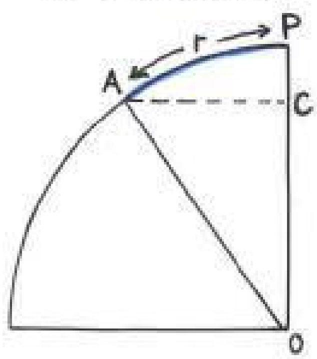
Sphere of radius a

Geodesics are great circles

Arc length $AP = BP = r$

Angle $A\hat{P}B = \theta$

In cross-section



$$\frac{A\hat{O}C}{\frac{\pi}{2}} = \frac{r}{\frac{2\pi a}{4}} \Rightarrow A\hat{O}C = \frac{r}{a}$$

$$AC = OA \sin A\hat{O}C$$

$$= a \sin \frac{r}{a}$$

$$\text{Circumference, } C = 2\pi a \sin\frac{r}{a} = 2\pi a\left[\frac{r}{a} - \frac{r^3}{6a^3} + \ldots\right]$$

$$= 2\pi r - \frac{\pi r^3}{3a^2} + \ldots$$

$$K = \frac{3}{\pi} \lim_{r\to 0} \frac{2\pi r - C}{r^3} = \frac{1}{a^2}$$

Note that K is measurable LOCALLY, but is very small on the surface of the Earth.

Spacetime near an isolated mass – SCHWARZSCHILD METRIC

$$ds^2 = c^2\left(1-\frac{2GM}{c^2}\right)dt^2 - \left(\frac{dr^2}{1-\frac{2GM}{c^2}} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right)$$

Interval between EVENTS, with coordinates (t, r, θ, ϕ) and $(t+dt, r+dr, \theta+d\theta, \phi+d\phi)$ in the spacetime close to an isolated point mass, M

We can use this metric to study some of the classical tests of GR :-

- Gravitational bending of light
- Advance of perihelion of Mercury
- Clocks in a gravitational field

Black Holes

Consider again the Schwarzschild metric :-

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2}\right)dt^2 - \left(\frac{dr^2}{1 - \frac{2GM}{c^2}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right)$$

Note that, when $r = \frac{2GM}{c^2}$ coeff of $dr^2 \to \infty$

Usually $r_s = \frac{2GM}{c^2}$ is much smaller than the actual radius of the "point" mass, M

e.g. Sun : $r_s \simeq 3\,\text{km}$ Earth : $r_s \simeq 9\,\text{mm}$

We call r_s the SCHWARZSCHILD RADIUS

For stars with $M \gtrsim 3M_\odot$ end state of stellar evolution is irreversible collapse

$\Rightarrow$ $\boxed{R_* \text{ shrinks to within } r_s}$

$\Downarrow$

Any material particle or photon which travels inside r_s cannot escape

$\downarrow$

Formation of a black hole