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Eccentricity Generation in Hierarchical Triple Systems

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WHAT IS A HIERARCHICAL TRIPLE SYSTEM?

A hierarchical triple system consists of:

a binary system and a third body on a wider orbit.

The motion of such a system can be pictured as the motion of two binaries: the binary itself (inner binary) and the binary which consists of the third body and the centre of mass of the binary (outer binary).

A simple example: Sun-Earth + Jupiter.

The presence of the third body is going to affect the dynamics of the inner binary.

Our main interest: the inner binary eccentricity.

We concentrate on systems with well separated components, i.e. systems with large period ratio

$$X = \frac{P_2}{P_1} > 10.$$

For most hierarchical triple stars, $X \sim 100$ and these systems are probably very stable dynamically. However, there are systems with much smaller period ratios, like the system HD 109648 with $X = 22$ (Jha et al. 2000), the λ Tau system, with $X = 8.3$ (Fekel & Tomkin 1982) and the CH Cyg system with $X = 7.0$ (Hinkle et al. 1993)

Also, we consider the initial inner eccentricity to be be zero.

We start from the simplest case: initially circular and coplanar orbits.

We take four steps:

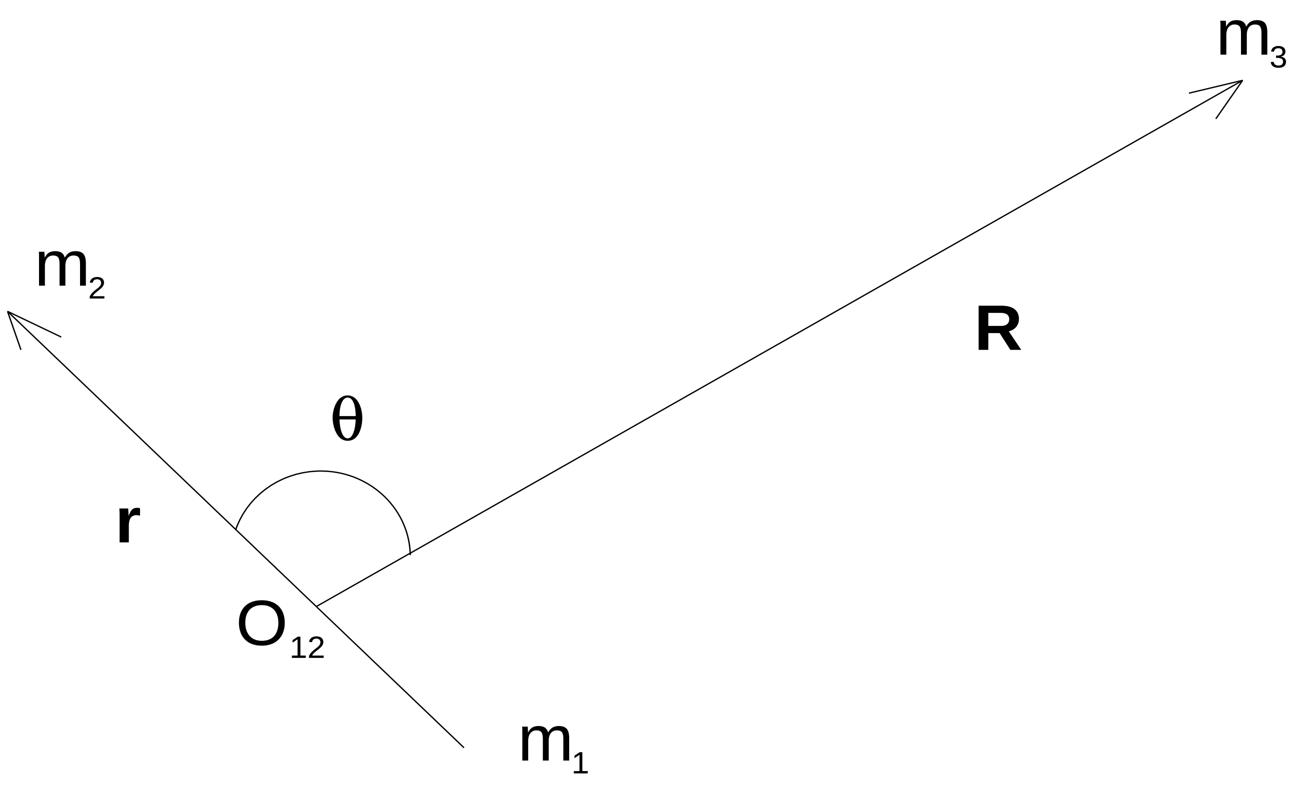
- Calculation of short period terms

- Calculation of secular terms

- Calculation of the initial outer secular eccentricity

- Combination of short-period and secular terms

The motion of the system can be studied using the Jacobi formulation:



SHORT PERIOD TERMS

Equation of motion of the inner binary:

$$\ddot{\mathbf{r}} = -G(m_1 + m_2)\frac{\mathbf{r}}{r^3} + \mathbf{F} \qquad (1)$$

$\mathbf{F}$, the perturbation to the inner binary motion

$$\mathbf{F} = Gm_3\left(\frac{\mathbf{R} - \mu_1\mathbf{r}}{|\mathbf{R} - \mu_1\mathbf{r}|^3} - \frac{\mathbf{R} + \mu_2\mathbf{r}}{|\mathbf{R} + \mu_2\mathbf{r}|^3}\right) =$$

$$= Gm_3\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{\mu_1|\mathbf{R} - \mu_1\mathbf{r}|} + \frac{1}{\mu_2|\mathbf{R} + \mu_2\mathbf{r}|}\right)$$

$$\mu_i = \frac{m_i}{m_1 + m_2}, \quad i = 1, 2.$$

Recall: third body at considerable distance, thus r/R small.

Then, the inverse distances in the previous equation can be expressed as:

$$\frac{1}{|\mathbf{R} - \mu_1 \mathbf{r}|} = \frac{1}{R} \sum_{n=o}^{\infty} \left(\frac{\mu_1 r}{R} \right)^n P_{\mathrm{n}}(\cos\theta)$$

and

$$\frac{1}{|\mathbf{R} + \mu_2 \mathbf{r}|} = \frac{1}{R} \sum_{n=o}^{\infty} \left(-\frac{\mu_2 r}{R} \right)^n P_{\mathrm{n}}(\cos\theta),$$

where P_{n} are Legendre polynomials

Expanding to third order, the perturbation becomes

$$\mathbf{F} = Gm_3 \frac{\partial}{\partial \mathbf{r}} \left(\frac{3}{2} \frac{(\mathbf{r} \cdot \mathbf{R})^2}{R^5} - \frac{1}{2} \frac{r^2}{R^3} - \right.$$

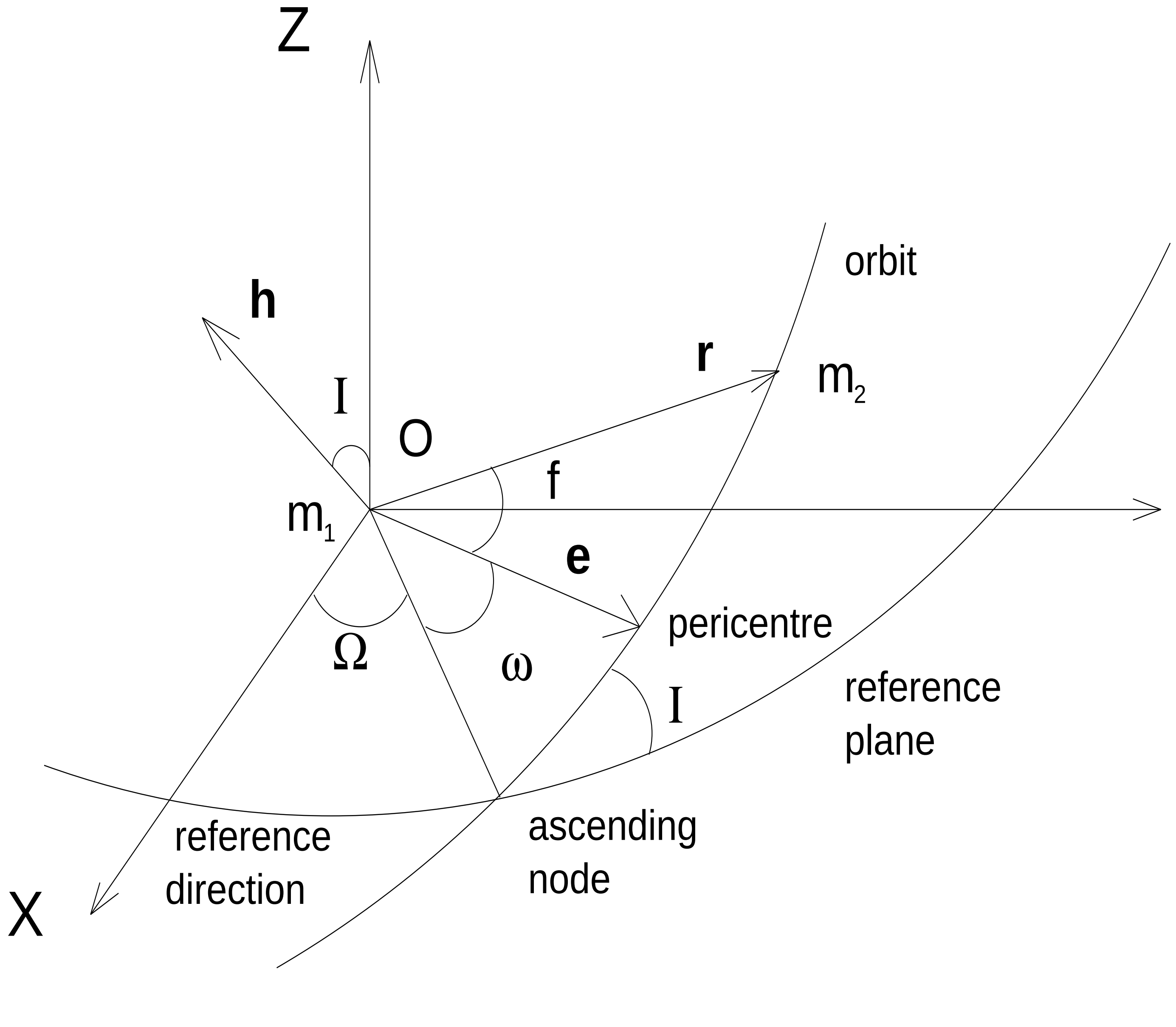
$$\left. - \frac{5(\mu_2^2 - \mu_1^2)}{2} \frac{(\mathbf{r} \cdot \mathbf{R})^3}{R^7} + \frac{3(\mu_2^2 - \mu_1^2)}{2} \frac{r^2 (\mathbf{r} \cdot \mathbf{R})}{R^5} \right)$$

The first two terms in the above equation come from the quadrupole term (P_2), while the other two come from the octupole term (P_3).

We now make use of the Runge-Lenz vector, i.e. the vector which has the same direction as the radius vector to the pericentre and whose magnitude is equal to the eccentricity of the orbit.

$$\mathbf{e}_1 = -\frac{\mathbf{r}}{r} + \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu}, \tag{2}$$

where $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ and $\mu = G(m_1 + m_2)$.


Z
h
I
O
r
m2
orbit
f
m1
e
pericentre
Ω
ω
I
reference
plane
ascending
node
reference
direction
X

Differentiating equation (2) and substituting for $\mathbf{F}$ (neglecting the term $\mathbf{r}\cdot\dot{\mathbf{r}}$), we obtain:

$$\dot{\mathbf{e}}_1 = \frac{Gm_3}{\mu R^3}\left[\left(6\frac{(\mathbf{r}\cdot\mathbf{R})(\dot{\mathbf{r}}\cdot\mathbf{R})}{R^2} - 15(\mu_2^2-\mu_1^2)\frac{(\mathbf{r}\cdot\mathbf{R})^2(\dot{\mathbf{r}}\cdot\mathbf{R})}{R^4} + \right.\right.$$
$$+3(\mu_2^2-\mu_1^2)\frac{r^2(\dot{\mathbf{r}}\cdot\mathbf{R})}{R^2}\right)\mathbf{r} + \left(r^2 - 3\frac{(\mathbf{r}\cdot\mathbf{R})^2}{R^2} + \right.$$
$$\left.\left. +\frac{15}{2}(\mu_2^2-\mu_1^2)\frac{(\mathbf{r}\cdot\mathbf{R})^3}{R^4} - -\frac{9}{2}(\mu_2^2-\mu_1^2)\frac{r^2(\mathbf{r}\cdot\mathbf{R})}{R^2}\right)\dot{\mathbf{r}}\right] \quad (3)$$

Now, the Jacobi vectors can be represented approximately in polar form as

$$\mathbf{r} = a_1(\cos n_1 t, \sin n_1 t)$$

and

$$\mathbf{R} = a_2(\cos(n_2 t + \phi), \sin(n_2 t + \phi))$$

After substituting in (3) and integrating, the components x_1 and y_1 of the eccentric vector become (expanding in powers of $\frac{1}{X}$ and retaining the two leading terms):

$$x_1 = \frac{m_3}{M}\frac{1}{X^2}(P_{\mathrm{x21}}(t) + X^{\frac{1}{3}}P_{\mathrm{x31}}(t)) + C_{\mathrm{x}_1} \quad (4)$$

$$y_1 = \frac{m_3}{M}\frac{1}{X^2}(P_{\mathrm{y21}}(t) + X^{\frac{1}{3}}P_{\mathrm{y31}}(t)) + C_{\mathrm{y}_1} \quad (5)$$

where

$$
\begin{aligned}
P_{\mathrm{x}21}(t) &= -\frac{1}{2}\cos n_1 t + \frac{1}{4}\cos\left((3n_1 - 2n_2)t - 2\phi\right) + \\
&\quad + \frac{9}{4}\cos\left((n_1 - 2n_2)t - 2\phi\right) \\
P_{\mathrm{x}31}(t) &= \frac{15}{16} m_* \cos\left(n_2 t + \phi\right) \\
P_{\mathrm{y}21}(t) &= -\frac{1}{2}\sin n_1 t + \frac{1}{4}\sin\left((3n_1 - 2n_2)t - 2\phi\right) - \\
&\quad - \frac{9}{4}\sin\left((n_1 - 2n_2)t - 2\phi\right) \\
P_{\mathrm{y}31}(t) &= \frac{15}{16} m_* \sin\left(n_2 t + \phi\right)
\end{aligned}
$$

and

$$
m_* = \frac{m_2 - m_1}{(m_1 + m_2)^{\frac{2}{3}} M^{\frac{1}{3}}}.
$$

M is the total mass of the system and C_{x_1} and C_{y_1} are constants of integration. The semi-major axes and mean motions were treated as constants in the above calculation.

SECULAR TERMS

Averaged Hamiltonian over the inner and outer orbital periods by means of the Von Zeipel method (e.g. Marchal 1990, Krymolowski & Mazeh 1999):

$$H = -\frac{Gm_1m_2}{2a_\mathrm{S}} - \frac{G(m_1+m_2)m_3}{2a_\mathrm{T}} + Q_1 + Q_2 + Q_3$$

where

$$Q_1 = -\frac{1}{8}\frac{Gm_1m_2m_3a_\mathrm{S}^2}{(m_1+m_2)a_\mathrm{T}^3(1-e_\mathrm{T}^2)^{\frac{3}{2}}}(2+3e_\mathrm{S}^2),$$

$$Q_2 = \frac{15 G m_1 m_2 m_3 (m_1 - m_2) a_S^3 e_S e_T}{64 (m_1 + m_2)^2 a_T^4 (1 - e_T^2)^{\frac{5}{2}}} \times$$

$$\times \cos(g_S - g_T)(4 + 3e_S^2),$$

$$Q_3 = -\frac{15}{64} \frac{G m_1 m_2 m_3^2 a_S^{\frac{7}{2}} e_S^2 (1 - e_S^2)^{\frac{1}{2}}}{(m_1 + m_2)^{\frac{3}{2}} M^{\frac{1}{2}} a_T^{\frac{9}{2}} (1 - e_T^2)^3} \times$$

$$\times [5(3 + 2e_T^2) + 3e_T^2 \cos 2(g_S - g_T)].$$

Subscript S refers to the inner long period orbit

Subscript T refers to the outer long period orbit

g denotes longitude of pericentre

Interaction between the two binaries:

Q_1 (P_2 Legendre polynomial)

Q_2 (P_3 Legendre polynomial)

Q_3 (arises from the canonical transformation).

From Hamilton's equations:

$$\frac{\mathrm{d}x_{\mathsf{S}}}{\mathrm{d}\tau} = \frac{5}{16}\alpha\frac{e_{\mathsf{T}}}{(1-e_{\mathsf{T}}^2)^{\frac{5}{2}}}(1-e_{\mathsf{S}}^2)^{\frac{1}{2}}[(4+3e_{\mathsf{S}}^2)\sin g_{\mathsf{T}}+$$

$$+6(x_{\mathsf{S}}y_{\mathsf{S}}\cos g_{\mathsf{T}}+y_{\mathsf{S}}^2\sin g_{\mathsf{T}})]-[\frac{(1-e_{\mathsf{S}}^2)^{\frac{1}{2}}}{(1-e_{\mathsf{T}}^2)^{\frac{3}{2}}}+$$

$$+\frac{25}{8}\gamma\frac{3+2e_{\mathsf{T}}^2}{(1-e_{\mathsf{T}}^2)^3}(1-\frac{3}{2}e_{\mathsf{S}}^2)]y_{\mathsf{S}}+\frac{15}{8}\gamma\frac{e_{\mathsf{T}}^2}{(1-e_{\mathsf{T}}^2)^3}[y_{\mathsf{S}}\cos 2g_{\mathsf{T}}-$$

$$-x_{\mathsf{S}}\sin 2g_{\mathsf{T}}-\frac{y_{\mathsf{S}}}{2}(x_{\mathsf{S}}^2+3y_{\mathsf{S}}^2)\cos 2g_{\mathsf{T}}+x_{\mathsf{S}}(x_{\mathsf{S}}^2+2y_{\mathsf{S}}^2)\sin 2g_{\mathsf{T}}]$$

$$\frac{\mathrm{d}y_{\mathsf{S}}}{\mathrm{d}\tau} = -\frac{5}{16}\alpha\frac{e_{\mathsf{T}}}{(1-e_{\mathsf{T}}^2)^{\frac{5}{2}}}(1-e_{\mathsf{S}}^2)^{\frac{1}{2}}[(4+3e_{\mathsf{S}}^2)\cos g_{\mathsf{T}}+$$

$$+6(x_{\mathsf{S}}y_{\mathsf{S}}\sin g_{\mathsf{T}}+x_{\mathsf{S}}^2\cos g_{\mathsf{T}})]+[\frac{(1-e_{\mathsf{S}}^2)^{\frac{1}{2}}}{(1-e_{\mathsf{T}}^2)^{\frac{3}{2}}}+$$

$$+\frac{25}{8}\gamma\frac{3+2e_{\mathsf{T}}^2}{(1-e_{\mathsf{T}}^2)^3}(1-\frac{3}{2}e_{\mathsf{S}}^2)]x_{\mathsf{S}}+\frac{15}{8}\gamma\frac{e_{\mathsf{T}}^2}{(1-e_{\mathsf{T}}^2)^3}[x_{\mathsf{S}}\cos 2g_{\mathsf{T}}+$$

$$+y_{\mathsf{S}}\sin 2g_{\mathsf{T}}-\frac{x_{\mathsf{S}}}{2}(y_{\mathsf{S}}^2+3x_{\mathsf{S}}^2)\cos 2g_{\mathsf{T}}-y_{\mathsf{S}}(y_{\mathsf{S}}^2+2x_{\mathsf{S}}^2)\sin 2g_{\mathsf{T}}]$$

$$\frac{\mathrm{d}g_{\mathsf{T}}}{\mathrm{d}\tau} = \frac{\beta(2+3e_{\mathsf{S}}^2)}{2(1-e_{\mathsf{T}}^2)^2} - \frac{5}{16}\frac{\alpha\beta(1+4e_{\mathsf{T}}^2)}{e_{\mathsf{T}}(1-e_{\mathsf{T}}^2)^3}(4+3e_{\mathsf{S}}^2)\times$$

$$\times(x_{\mathsf{S}}\cos g_{\mathsf{T}} + y_{\mathsf{S}}\sin g_{\mathsf{T}}) + \frac{5}{8}\beta\gamma\times$$

$$\times\frac{(1-e_{\mathsf{S}}^2)^{\frac{1}{2}}}{(1-e_{\mathsf{T}}^2)^{\frac{7}{2}}}[5e_{\mathsf{S}}^2(11+4e_{\mathsf{T}}^2) + 3(1+2e_{\mathsf{T}}^2)\times$$

$$\times((x_{\mathsf{S}}^2 - y_{\mathsf{S}}^2)\cos 2g_{\mathsf{T}} + 2x_{\mathsf{S}}y_{\mathsf{S}}\sin 2g_{\mathsf{T}})]$$

$$\frac{\mathrm{d}e_{\mathsf{T}}}{\mathrm{d}\tau} = \frac{5}{16}\frac{\alpha\beta}{(1-e_{\mathsf{T}}^2)^2}(4+3e_{\mathsf{S}}^2)(y_{\mathsf{S}}\cos g_{\mathsf{T}} - x_{\mathsf{S}}\sin g_{\mathsf{T}}) -$$

$$-\frac{15}{8}\beta\gamma\frac{e_{\mathsf{T}}(1-e_{\mathsf{S}}^2)^{\frac{1}{2}}}{(1-e_{\mathsf{T}}^2)^{\frac{5}{2}}}\times(2x_{\mathsf{S}}y_{\mathsf{S}}\cos 2g_{\mathsf{T}} - (x_{\mathsf{S}}^2 - y_{\mathsf{S}}^2)\sin 2g_{\mathsf{T}})$$

where

$$x_{\mathsf{S}} = e_{\mathsf{S}}\cos g_{\mathsf{S}}, \quad y_{\mathsf{S}} = e_{\mathsf{S}}\sin g_{\mathsf{S}},$$

$$\alpha = \frac{m_1 - m_2}{m_1 + m_2}\frac{a_S}{a_T}, \ \beta = \frac{m_1 m_2 M^{\frac{1}{2}}}{m_3(m_1 + m_2)^{\frac{3}{2}}}(\frac{a_S}{a_T})^{\frac{1}{2}},$$

$$\gamma = \frac{m_3}{M^{\frac{1}{2}}(m_1 + m_2)^{\frac{1}{2}}}(\frac{a_S}{a_T})^{\frac{3}{2}}$$

and

$$\mathrm{d}\tau = \frac{3}{4}\frac{G^{\frac{1}{2}} m_3 a_S^{\frac{3}{2}}}{a_T^3 (m_1 + m_2)^{\frac{1}{2}}}\mathrm{d}t.$$

The system of d.e. reduces to one that can be solved analytically:

(neglecting terms of $O(e_{\mathsf{S}}^2)$ and $O(e_{\mathsf{T}}^2)$,considering e_{T} constant and keeping the dominant term in $\mathrm{d}g_{\mathsf{T}}/\mathrm{d}\tau$)

$$\begin{aligned} \frac{\mathrm{d}x_{\mathsf{S}}}{\mathrm{d}\tau} &= -By_{\mathsf{S}} + C\sin g_{\mathsf{T}} \\ \frac{\mathrm{d}y_{\mathsf{S}}}{\mathrm{d}\tau} &= Bx_{\mathsf{S}} - C\cos g_{\mathsf{T}} \\ \frac{\mathrm{d}g_{\mathsf{T}}}{\mathrm{d}\tau} &= A, \end{aligned} \qquad (6)$$

where

$$A = \beta,\ \ B = 1 + \frac{75}{8}\gamma,\ \ C = \frac{5}{4}\alpha e_{\mathsf{T}}.$$

The solution to system (6) is:

$$x_{\mathsf{S}}(\tau) = K_1 \cos B\tau + K_2 \sin B\tau + \frac{C}{B-A}\cos(A\tau + g_{\mathsf{T}_0}) \qquad (7)$$

$$y_{\mathsf{S}}(\tau) = K_1 \sin B\tau - K_2 \cos B\tau + \frac{C}{B-A}\sin(A\tau + g_{\mathsf{T}_0}) \qquad (8)$$

where K_1, K_2 are constants of integration and $g_{\mathsf{T}0}$ is the initial value of g_{T}.

INITIAL OUTER SECULAR ECCENTRICITY

$$e_{out} = e_2 + e_T \qquad (9)$$

To calculate e_2 (e_T constant), we follow the same procedure as we did for the inner orbit (eccentric vector etc.) That leads to:

$$x_2 = \frac{3}{4}\frac{m_1 m_2}{(m_1+m_2)^{\frac{4}{3}} M^{\frac{2}{3}}}\frac{1}{X^{\frac{4}{3}}}\cos(n_2 t + \phi) + C_{x_2}$$

$$y_2 = \frac{3}{4}\frac{m_1 m_2}{(m_1+m_2)^{\frac{4}{3}} M^{\frac{2}{3}}}\frac{1}{X^{\frac{4}{3}}}\sin(n_2 t + \phi) + C_{y_2}$$

Since e_{out} is initially zero:

$$e_T = \frac{3}{4}\frac{m_1 m_2}{(m_1+m_2)^{\frac{4}{3}} M^{\frac{2}{3}}}\frac{1}{X^{\frac{4}{3}}}.$$

INNER ECCENTRICITY FORMULA

Combining the short period and secular terms:

$$x_{\text{in}} = x_1 - C_{\text{x}_1} + x_{\text{S}} \tag{10}$$

$$y_{\text{in}} = y_1 - C_{\text{y}_1} + y_{\text{S}} \tag{11}$$

and averaging over time and over the initial relative phase ϕ, we get:

$$\overline{e_{\text{in}}^2} =< x_{\text{in}}^2 + y_{\text{in}}^2 >= \frac{m_3^2}{M^2}\frac{1}{X^4}\left(\frac{43}{4} + \frac{225}{128}m_*^2 X^{\frac{2}{3}}\right)+$$
$$+\frac{15}{8}\frac{m_3}{M}\frac{m_*}{X^{\frac{5}{3}}}\frac{C}{A-B} + 2\left(\frac{C}{A-B}\right)^2. \tag{12}$$

COMPARISON WITH OTHER RESULTS

Eggleton and collaborators (e.g. Eggleton & Kiseleva (1996)), based on results from numerical integrations of coplanar, prograde and initially circular orbits, derived the following empirical formula for the inner mean eccentricity:

$$\bar{e}_{\text{in}} = \frac{A}{X^{1.5}\sqrt{X-B}}, \tag{13}$$

where A and B depend on the mass ratios. For three equal masses

$$A = 1.167 \qquad \text{and} \qquad B = 3.814.$$

Equation (13) can be expanded to first order in terms of $\frac{1}{X}$, yielding

$$\bar{e}_{\rm in} = \frac{1.167}{X^2}(1 + \frac{3.814}{2X}). \tag{14}$$

Our way of calculating the eccentricity, for the case of three equal masses yields:

$$\bar{e}_{\rm in} = \frac{1.157}{X^2}(1 + \frac{3.816}{2X}). \tag{15}$$

NUMERICAL TESTING

We integrated HTS using a symplectic integrator with time transformation

(Mikkola 1997).

SHORT PERIOD

The following table shows the percentage error between the averaged numerical and averaged theoretical e_{in} (short period) . The theoretical model is based on equations (4) and (5). For all systems, $\phi = 90^{\circ}$.

Integration time: one outer orbital period.

For each pair (m_3, X_0), there are five entries, corresponding, from top to bottom, to the following inner binaries:

$m_1 = 0.1 - m_2 = 0.9$,

$m_1 = 0.2 - m_2 = 0.8$,

$m_1 = 0.3 - m_2 = 0.7$,

$m_1 = 0.4 - m_2 = 0.6$ and

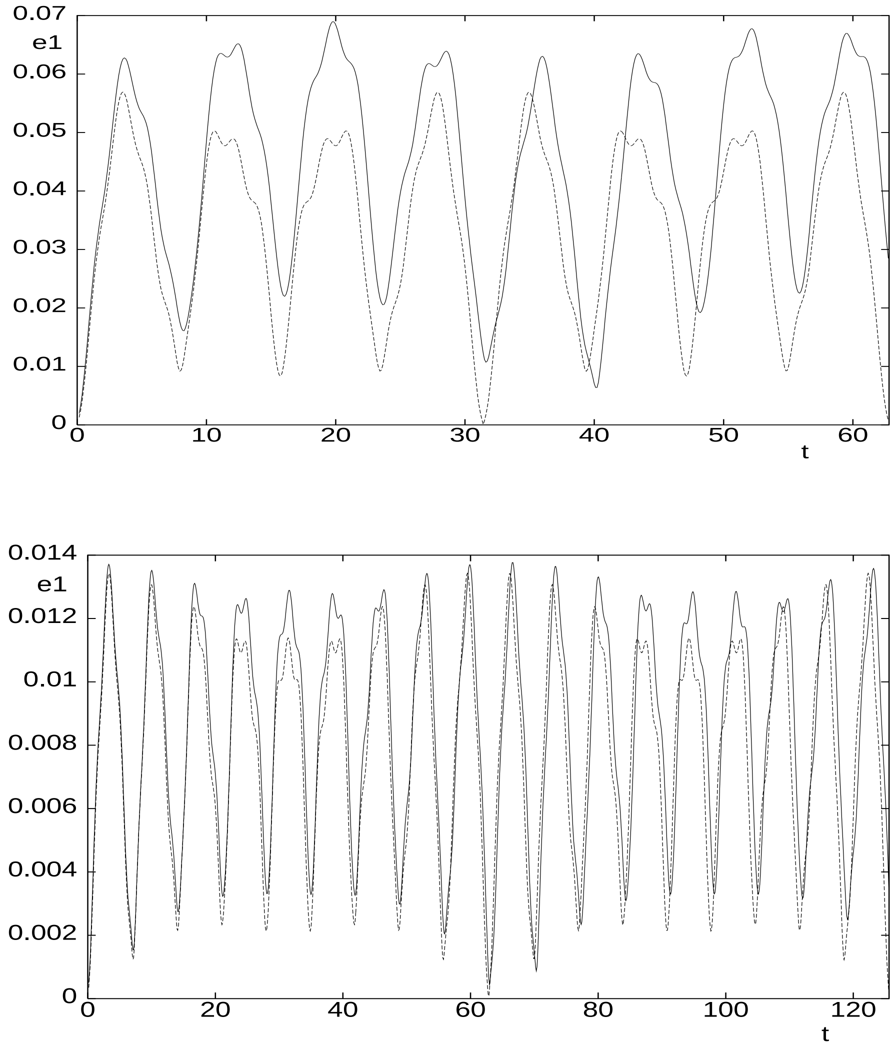
$m_1 = 0.5 - m_2 = 0.5$.

A dash in the Table denotes that the analogy among the masses was outside the range $10 : 1$.

$m_3\backslash\ X_0$	10	15	20	25	30	50
0.05	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	6.2	2.9	1.7	1.2	0.9	0.4
0.09	18.6	11.7	8.5	6.6	5.4	3
	19.3	12.3	8.9	6.9	5.7	3.2
	19.8	12.7	9.3	7.3	6	3.4
	20.1	13	9.5	7.5	6.2	3.6
	6.8	3.4	2.1	1.4	1.1	0.5
0.5	24.1	15.5	11.3	8.9	7.3	4.3
	24.5	15.7	11.5	9.1	7.4	4.3
	24.6	15.8	11.6	9.2	7.5	4.4
	24.6	15.9	11.7	9.2	7.6	4.4
	12	6.5	4.3	3.2	2.5	1.3
1	27.9	18	13.2	10.4	8.6	5
	28	18	13.2	10.4	8.6	5
	28	18	13.1	10.4	8.6	5
	27.8	17.9	13.1	10.3	8.5	5
	15.8	8.8	5.9	4.4	3.4	1.8
1.5	-	-	-	-	-	-
	30.1	19.4	14.2	11.2	9.2	5.4
	30	19.3	14.1	11.1	9.2	5.4
	29.8	19.1	14	11	9.1	5.3
	18.2	10.1	6.9	5.1	4	2.2
2	-	-	-	-	-	-
	31.6	20.3	14.9	11.7	9.6	5.7
	31.3	20.1	14.7	11.6	9.5	5.6
	31.1	20	14.6	11.5	9.4	5.5
	19.8	11.1	7.5	5.6	4.4	2.4

$m_3\backslash\ X_0$	10	15	20	25	30	50
2.6	-	-	-	-	-	-
	-	-	-	-	-	-
	32.4	20.8	15.2	12	9.9	5.8
	32.4	20.7	15.1	11.9	9.8	5.7
	21.3	11.9	8	6	4.7	2.5
3	-	-	-	-	-	-
	-	-	-	-	-	-
	33	21.2	15.5	12.1	10	5.9
	33	21.1	15.4	12.1	9.9	5.8
	22	12.3	8.3	6.2	5	2.6
3.4	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	33.5	21.4	15.6	12.2	10	5.8
	22.6	12.6	8.5	6.4	5	2.7
4	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	34	21.7	15.8	12.4	10.2	5.9
	23.3	13	8.8	6.6	5.2	2.8
4.5	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	23.8	13.3	9	6.7	5.3	2.9
5	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	24	13.5	9.1	6.8	5.4	2.9

Inner eccentricity against time. $m_1 = 0.5$, $m_3 = 5$, $X_0 = 10$ and $\phi = 90°$. Integration time span: ($T_{\text{out}} = 62.8$). Continuous-numerical integration, dashed curve-plot of equations (4),(5). Inner binary period: $T_{\text{in}} = 2\pi$. Top graph: $X_0 = 10$. Bottom graph: $X_0 = 20$.



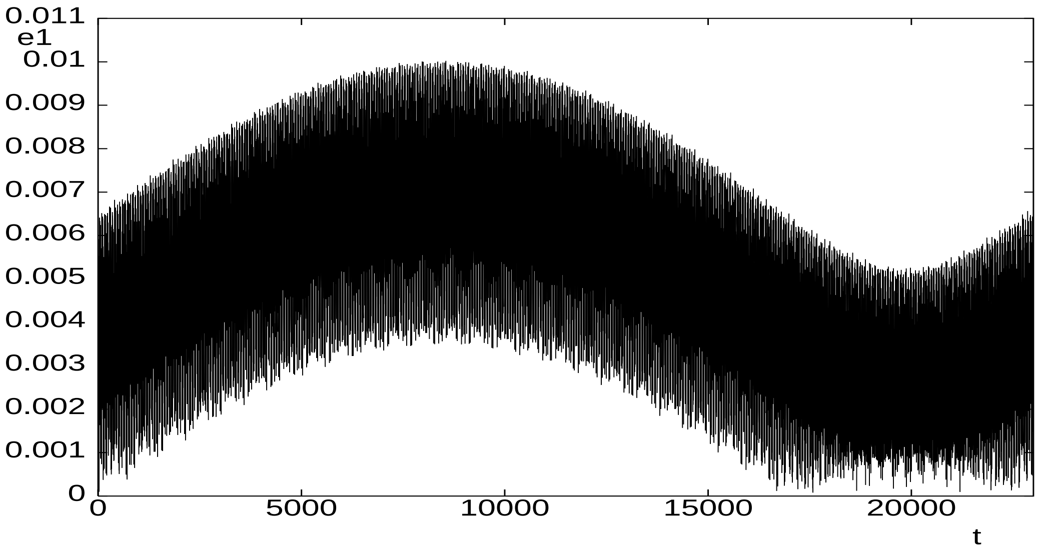
SHORT AND LONG PERIOD EFFECTS

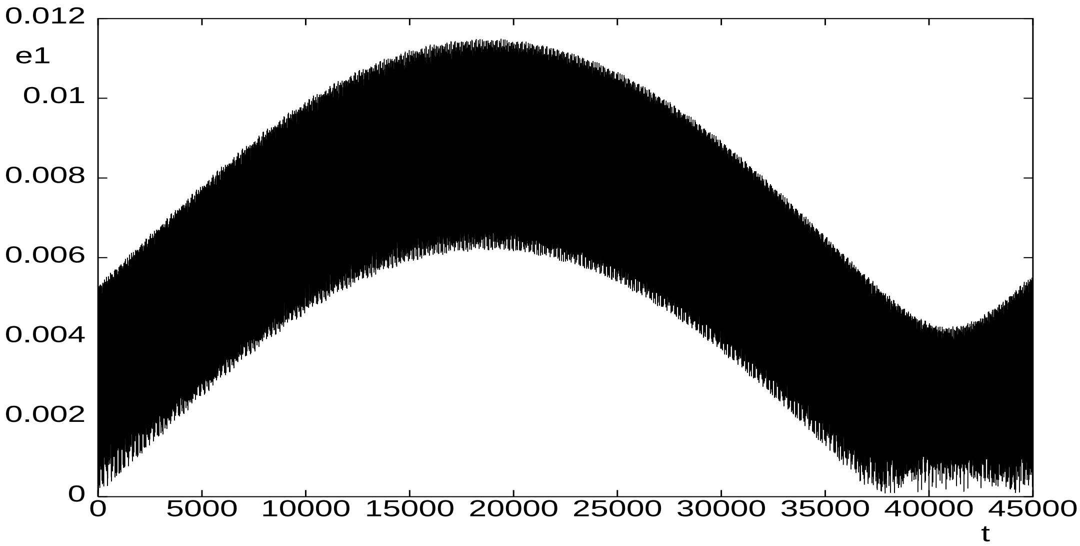
The absolute percentage error between the averaged, over time and initial phase ϕ, numerical $e_{\rm in}^2$ and equation (12).

Each system was numerically integrated for $\phi = 0^\circ - 360^\circ$ with a step of 10°. All the results are for $m_1 = 0.2$-$m_2 = 0.8$.

$m_3 \backslash\ X_0$	10	15	20	25	30	50
0.09	72.5	16.5	3	0.5	2.7	1.9
	23000	57000	97000	145000	196000	49000
0.5	37.1	25	18.7	14.6	12.2	6.9
		6000	10000	17000	23000	70000
1	41.5	27.7	20.5	16.9	13.8	8.2
				7500	15000	40000
1.5	43.9	29.5	22	17.4	14.3	8.1
2	45.4	30.5	22.9	18.2	15.1	8.7

$m_1 = 0.2$, $m_1 = 0.8$,$m_3 = 0.09$, $X_0 = 10$ and $\phi = 90°$, $T_{\text{in}} = 2\pi$ and $T_{\text{out}} = 62.8$. Top graph: numerical integrations, bottom graph: theoretical model.





COPLANAR, ECCENTRIC OUTER BINARY CASE:

We proceed similarly to the circular case.

A couple of problems:

i) In the expressions for $P_{\text{x31}}(t)$ and $P_{\text{y31}}(t)$, terms proportional to the true anomaly f appeared. To eliminate f from our expressions, we used the following series expansion (Murray & Dermott 1999)

$$f = l + 2e \sin f - \frac{3}{4}e^2 \sin 2f + \frac{1}{3}e^3 \sin 3f + O(e^4),$$

where l is the mean anomaly, i.e. we replaced f with the periodic part of the above equation.

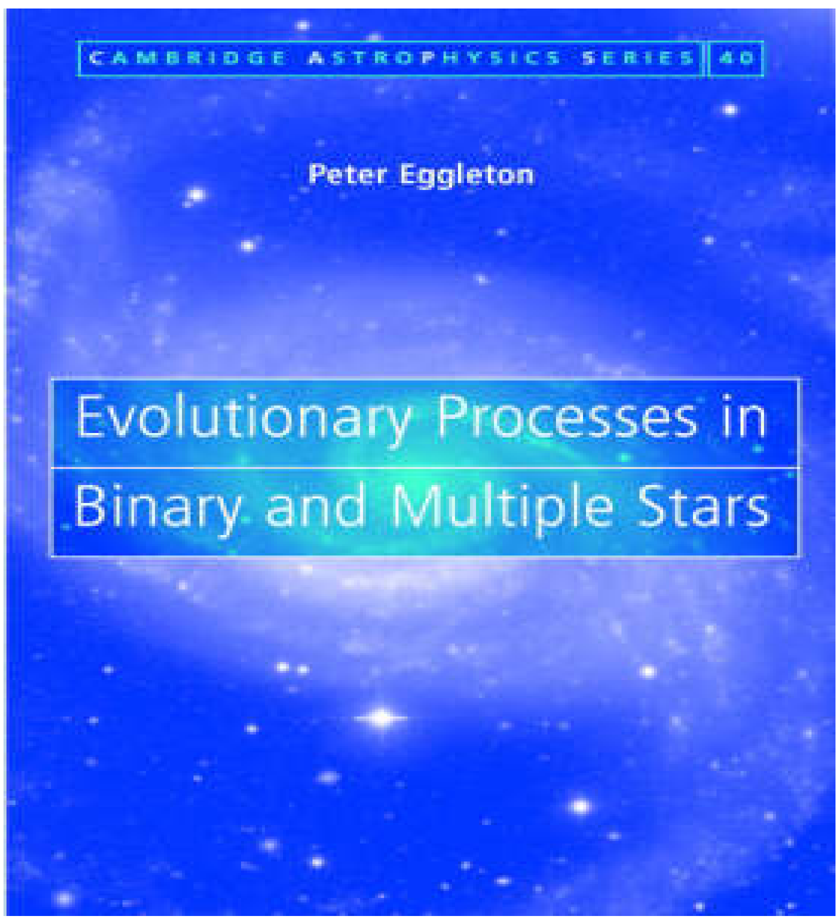
ii) In order to average the P_{31} term over f , we chose to expand $(1 + e \cos f)^{-2}$ binomially including terms up to $O(e^3)$.

For more details: Georgakarakos (2003)

The final formula for this case:

$$\overline{e_{\text{in}}^2} = \frac{m_3^2}{M^2}\frac{1}{X^4(1-e^2)^{\frac{9}{2}}}\Big[\frac{43}{8}+\frac{129}{8}e^2+\frac{129}{64}e^4+\frac{1}{(1-e^2)^{\frac{3}{2}}}(\frac{43}{8}+\frac{645}{16}e^2+ +\frac{1935}{64}e^4+\frac{215}{128}e^6)+\frac{1}{X^2(1-e^2)^3}[\frac{365}{18}+\frac{44327}{144}e^2+\frac{119435}{192}e^4+ +\frac{256105}{1152}e^6+\frac{68335}{9216}e^8+\frac{1}{(1-e^2)^{\frac{3}{2}}}(\frac{365}{18}+\frac{7683}{16}e^2+\frac{28231}{16}e^4 +\frac{295715}{192}e^6+\frac{2415}{8}e^8+\frac{12901}{2048}e^{10})]+\frac{1}{X(1-e^2)^{\frac{3}{2}}}[\frac{61}{3}+\frac{305}{2}e^2 +\frac{915}{8}e^4+\frac{305}{48}e^6+\frac{1}{(1-e^2)^{\frac{3}{2}}}(\frac{61}{3}+\frac{854}{3}e^2+\frac{2135}{4}e^4+\frac{2135}{12}e^6+ +\frac{2135}{384}e^8)]+m_*^2X^{\frac{2}{3}}(1-e^2)[\frac{225}{256}+\frac{3375}{1024}e^2+\frac{7625}{2048}e^4+\frac{29225}{8192}e^6+ +\frac{48425}{16384}e^8+\frac{825}{2048}e^{10}+\frac{1}{(1-e^2)^{\frac{3}{2}}}(\frac{225}{256}+\frac{2925}{1024}e^2+\frac{775}{256}e^4+ +\frac{2225}{8192}e^6+\frac{25}{512}e^8)]+m_*^2\frac{1}{X^{\frac{4}{3}}(1-e^2)^2}[\frac{8361}{4096}+\frac{125415}{8192}e^2+ +\frac{376245}{32768}e^4+\frac{41805}{65536}e^6+\frac{1}{(1-e^2)^{\frac{3}{2}}}(\frac{8361}{4096}+\frac{58527}{2048}e^2+ +\frac{877905}{16384}e^4+\frac{292635}{16384}e^6+\frac{292635}{524288}e^8)]\Big]+2(\frac{C}{B-A})^2. \qquad (16)$$

Equation (16) to be incorporated in Peter Eggleton's numerical code for binary evolution.



NON-COPLANAR, CIRCULAR BINARIES CASE

only holds for $I_0 < 39.23^\circ$ or $I_0 > 140.77^\circ$ (due to the Kozai effect)

The final formula for this case (for more details Georgakarakos 2004)

$$\begin{aligned}
\overline{e_{\rm in}^2} \;=\; & < x_{\rm in}^2 + y_{\rm in}^2 >= \frac{m_3^2}{M^2}\frac{1}{X^4}\left[\frac{221}{64} - \frac{37}{32}\cos^2 I + \frac{369}{64}\cos^4 I + \right.\\
& + \frac{D}{B}(\frac{19}{16} - \frac{5}{2}\cos^2 I + 3\cos^4 I) + \frac{B}{D}\cos^2 I + \frac{1}{X^2}[\frac{1011}{144} + \\
& + \frac{471}{24}\cos^2 I + \frac{543}{144}\cos^4 I + \frac{49}{9}\frac{D}{B}\cos^2 I + \frac{B}{D}(\frac{121}{36} + \\
& + \frac{11}{9}\cos^2 I + \frac{1}{9}\cos^4 I)] + \frac{1}{X}[\frac{45}{3}\cos I + \frac{93}{6}\cos^3 I + \\
& + \frac{D}{B}(\frac{7}{6}\cos I + \frac{14}{3}\cos^3 I) + \frac{B}{D}(\frac{11}{3}\cos I + \frac{2}{3}\cos^3 I)] + \\
& + m_*^2 X^{\frac{2}{3}}[\frac{1275}{8192} + \frac{36525}{8192}\cos^2 I - \frac{103875}{8192}\cos^4 I + \frac{76875}{8192}\cos^6 I + \\
& + \frac{D}{B}(\frac{13925}{8192}\cos^2 I - \frac{18875}{4096}\cos^4 I + \frac{25625}{8192}\cos^6 I) + \frac{B}{D}(\frac{425}{8192} - \\
& - \frac{875}{4096}\cos^2 I + \frac{3125}{8192}\cos^4 I)] + \frac{m_*^2}{X^{\frac{4}{3}}}[\frac{138519}{131072} + \frac{62289}{131072}\cos^2 I +
\end{aligned}$$

$$+\frac{121185}{131072}\cos^4 I+\frac{102375}{131072}\cos^6 I+\frac{D}{B}(\frac{54333}{131072}-\frac{42435}{65536}\cos^2 I+$$
$$+\frac{119025}{131072}\cos^4 I)+\frac{B}{D}(\frac{94113}{131072}\cos^2 I-\frac{17955}{65536}\cos^4 I+$$
$$+\frac{10125}{131072}\cos^6 I)]+\frac{m_*^2}{X^{\frac{1}{3}}}[\frac{12495}{8192}\cos I-\frac{19875}{4096}\cos^3 I+$$
$$+\frac{24375}{8192}\cos^5 I+\frac{D}{B}(\frac{25545}{16384}\cos I-\frac{33975}{8192}\cos^3 I+\frac{43125}{16384}\cos^5 I)+$$
$$+\frac{B}{D}(-\frac{555}{16384}\cos I-\frac{5775}{8192}\cos^3 I+\frac{5625}{16384}\cos^5 I)]\Big]-\frac{m_3 m_* M_*}{MX^3}\times$$
$$\times[(\frac{335}{1024}\cos I-\frac{875}{512}\cos I^3+\frac{1775}{1024}\cos I^5)(1+\frac{D}{B})\frac{AC+BE}{BD-A^2}+$$
$$+(\frac{155}{1024}\cos I-\frac{335}{512}\cos I^3+\frac{875}{1024}\cos I^5)(1+\frac{B}{D})\frac{AE+CD}{BD-A^2}]-$$
$$-\frac{m_3 m_* M_*}{MX^4}[(\frac{219}{4096}-\frac{1935}{2048}\cos^2 I+\frac{3795}{4096}\cos^4 I)(1+\frac{D}{B})\times$$
$$\times\frac{AC+BE}{BD-A^2}+(\frac{687}{4096}\cos^2 I-\frac{1779}{2048}\cos^4 I+\frac{1575}{4096}\cos^6 I)\times$$
$$\times(1+\frac{B}{D})\frac{AE+CD}{BD-A^2}]+\frac{M_*^2}{X^{\frac{8}{3}}}[\frac{(AC+BE)^2}{(BD-A^2)^2}(1+\frac{D}{B})(\frac{29}{512}-$$
$$-\frac{47}{256}\cos^2 I+\frac{137}{512}\cos^4 I)+\frac{(AE+CD)^2}{(BD-A^2)^2}(1+\frac{B}{D})\times$$
$$\times(\frac{65}{512}\cos^2 I-\frac{119}{256}\cos^4 I+\frac{245}{512}\cos^6 I)]+\frac{1}{2}\frac{M_*^2}{X^{\frac{8}{3}}}\times$$
$$\times\left[\frac{(AC+BE)^2}{(BD-A^2)^2}+\frac{(AE+CD)^2}{(BD-A^2)^2}\right](\frac{29}{256}-\frac{29}{256}\cos^2 I-$$
$$-\frac{101}{256}\cos^4 I+\frac{245}{256}\cos^6 I).$$

The coplanar formulae (4 terms) have been also tested for planetary mass ratios (Georgakarakos 2006).

Initial aim: mass ratios $\sim 10^{-6} - 10^{6}$

Achieved: mass ratios $\sim 10^{-3} - 10^{3}$ due to computational time.

FUTURE AIMS

- 3D case with eccentric outer binary

- tidally evolving inner binary

- possible applications to exosolar systems

THANK YOU!