Mechanisms for the production of chaos in dynamical systems

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Why we study chaos

Chaos is important to understand the evolution of our (and others) solar system dynamics, because we find chaos in the orbits of the planets, of many asteroids, in the rotation of Mars and of satellites....

These lectures are an introduction to the basic phenomena which produce chaos in dynamical systems, which can be found in very simple examples and can be exported (with 'some' work....) to the much more complex systems of the real world.

Chaos in two dimensional maps

- Stable and unstable manifolds of saddle points
- Dynamics near the homoclinic tangles
- Barriers to diffusion: KAM curves and cantori
- Chaos and diffusion: etheroclinic orbits, Chirikov diffusion

Higher dimensional cases

- Arnold diffusion
- Chaos in the orbits of the giant planets of our solar system

What is chaos?

Chaos is related to the real possibility of predicting the future of a deterministic dynamical system

- If we could know exactly the laws of nature and the situation of the universe at the initial instant, we should be able to predict exactly the situation of this same universe at a subsequent instant. But even when the natural laws should have no further secret for us, we could know the initial situation only approximately. If that permits us to foresee the subsequent situation with the same degree of accuracy, ..., we say that the phenomenon has been predicted... But this is not always the case; it may happen that slight differences in the initial conditions produce very great differences in the final phenomena (H. Poincaré).

A model example

A way to visualize the differences in the orbits of close initial conditions is to represent the evolution in the phase–space of a set A of initial conditions. Let us denote by:

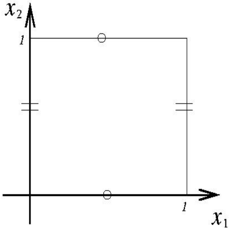
$$\psi : M \longrightarrow M \quad ,$$

a map such that, if $x \in M$ is the state of the system at time $t = 0$, $\psi(x)$ is the state of the system at the time 1. Therefore we represent the sequence:

$$A \;\; , \;\; \psi(A) \;\; , \;\; \psi^2(A) \;\; , \ldots \;\; , \psi^t(A) \;\; , \;\; \ldots$$

A model example of exponential divergence: the Arnold cat

- The phase–space M is $\mathbb{T}^2$



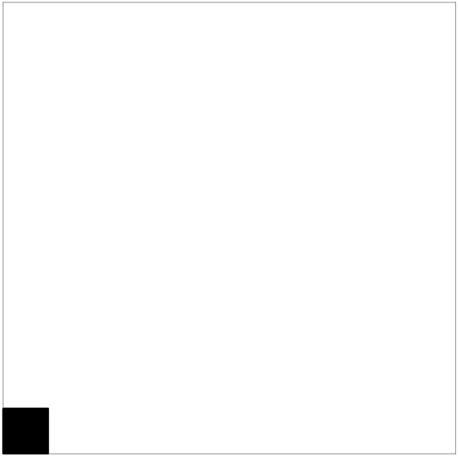
- The orbit of a point x is given by the iteration of the map

$$\psi : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$$

$$(x_1, x_2) \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} \mod (1)$$

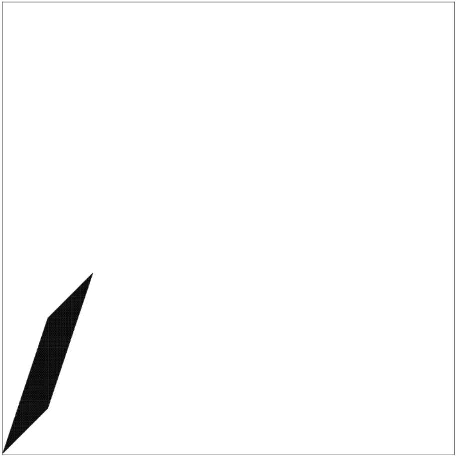
Dispersion of a square 0.1×0.1 in 6 iterations of the map

$t = 0$

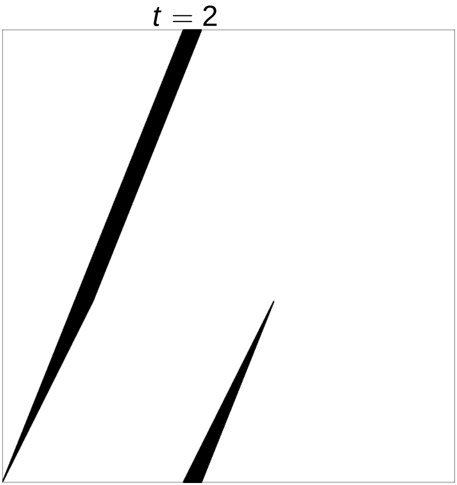


Dispersion of a square 0.1×0.1 in 6 iterations of the map

$t = 1$

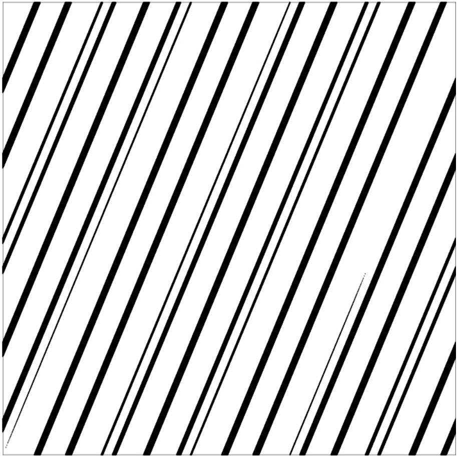


Dispersion of a square 0.1×0.1 in 6 iterations of the map



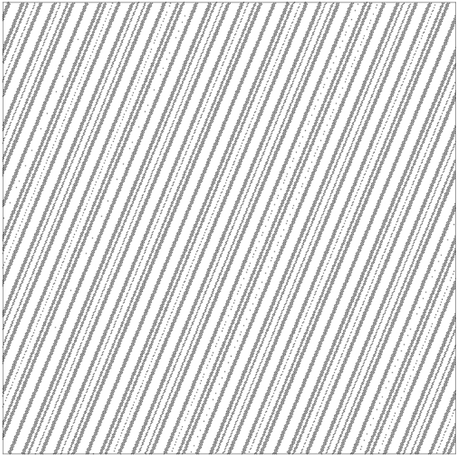
Dispersion of a square 0.1×0.1 in 6 iterations of the map

$t = 4$



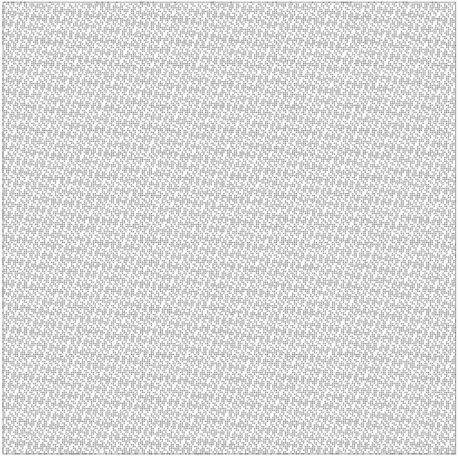
Dispersion of a square 0.1×0.1 in 6 iterations of the map

$t = 5$



Dispersion of a square 0.1×0.1 in 6 iterations of the map

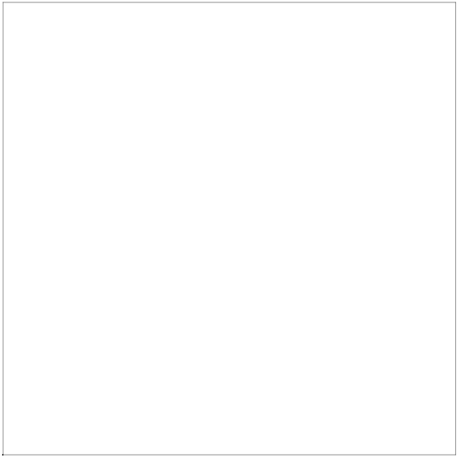
$t = 6$



- After 6 iterations of the map the square is spread in the entire phase–space, so that one is not able to distinguish the orbits of 2 points with initial conditions $\|z - z'\| \sim 0.1$.

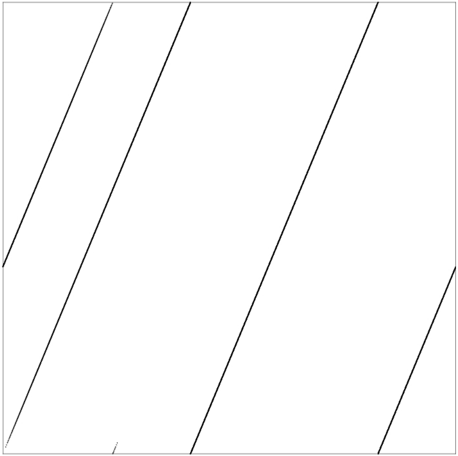
One could think that the “poor control” $T < 6$ on the orbits is due to the big distance 0.1 between initial conditions. We now use a square $10^{-14} \times 10^{-14}$ (smaller than atomic radii!).

$t = 20$



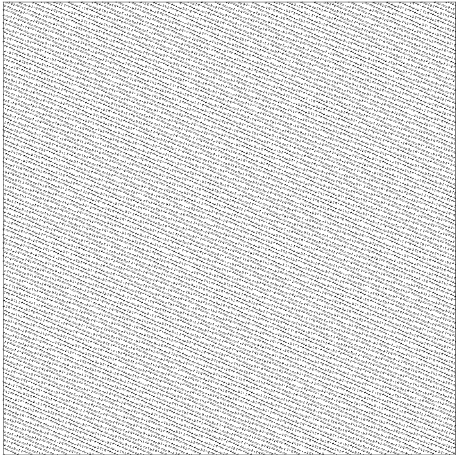
One could think that the "poor control" $T < 6$ on the orbits is due to the big distance 0.1 between initial conditions. We now use a square $10^{-14} \times 10^{-14}$ (smaller than atomic radii!).

$t = 27$



One could think that the “poor control” $T < 6$ on the orbits is due to the big distance 0.1 between initial conditions. We now use a square $10^{-14} \times 10^{-14}$ (smaller than atomic radii!).

$t = 31$



Ingredients which determine the chaotic behavior of the Arnold cat

- A fixed point:

$$\psi(0,0) = (0,0)$$

which is hyperbolic:

$$\frac{\partial \psi}{\partial x}(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} ,$$

has eigenvalues $0 < \lambda_1 = \frac{3-\sqrt{5}}{2} < 1 < \lambda_2 = \frac{3+\sqrt{5}}{2}$.

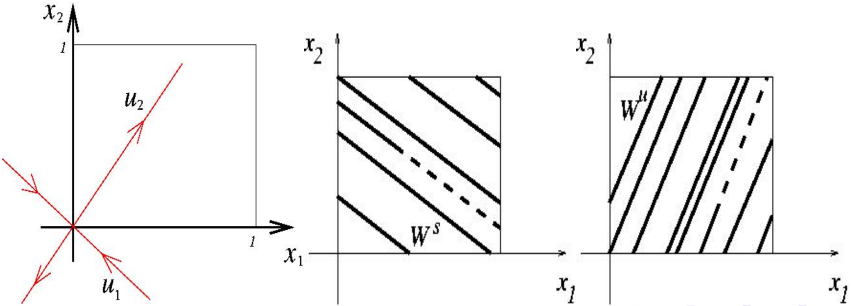
- The eigenvectors u_1, u_2 have irrational slopes: the stable set of the fixed point:

$$W^s = \{(x_1, x_2) : \lim_{t\to\infty} \psi^t(x_1, x_2) = (0,0)\} = < u_1 > \quad \text{mod } 1$$

and the unstable set of the fixed point:

$$W^u = \{(x_1, x_2) : \lim_{t\to\infty} \psi^{-t}(x) = (0,0)\} = < u_2 > \quad \text{mod } 1$$

are dense, with a dense set of transverse intersections.



The Arnold cat is constructed to exhibit the strongest possible chaotic behaviour:

- Fixed point with dense stable set W^s, unstable set W^u
- The linearized map is hyperbolic at any point

Even in more general cases the structure of the stable and unstable sets are the key to understand the chaotic behavior. In the generic case one needs:

- An hyperbolic fixed point
- An homoclinic intersection of W^s, W^u, i.e. a point of transverse intersection of the stable and unstable sets of the hyperbolic fixed point.

Some examples

Just to fix ideas with more general examples, we recall:

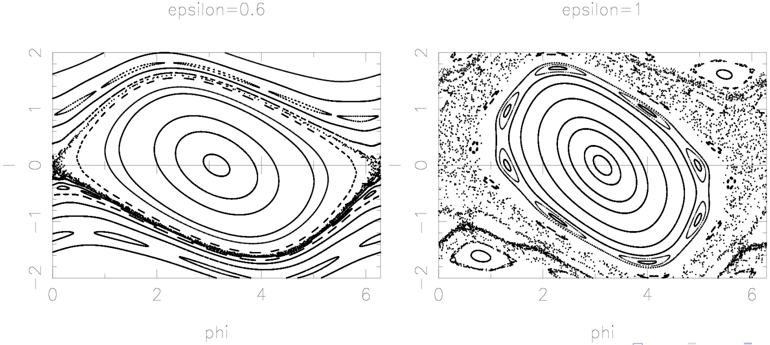
- the 2D standard map
- the rotation of a satellite
- normal forms of resonances of quasi-integrable systems (periodically perturbed pendulum)

Example: standard map

- The map is $(I, \varphi) \mapsto (I', \varphi')$ such that:

$$\begin{aligned} \varphi' &= \varphi + I \\ I' &= I + \varepsilon \sin(\varphi + I) \end{aligned} \tag{1}$$

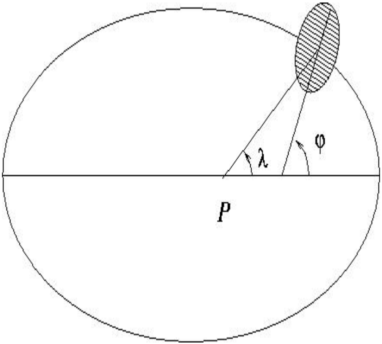
  with $(I, \varphi) \in \mathbb{R} \times \mathbb{T}$, $\varepsilon \in \mathbb{R}$ is a parameter.
- $(0, 0)$ is hyperbolic, $0 < \lambda_1 < 1 < \lambda_2$.



Example: Rotations of satellites

- A tri-axial satellite, whose CM moves on Keplerian orbit;
- The satellite is constrained to rotate in the plane x, y around an axis of inertia;
- The equation of motion for φ is:

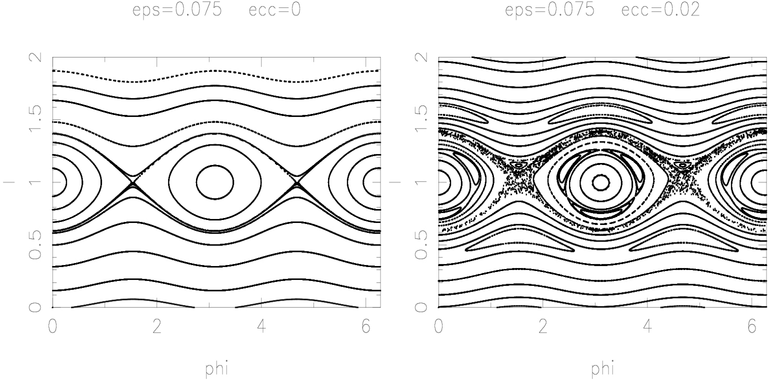
$$\ddot{\varphi} = -\frac{3}{2}\Omega^2 \left(\frac{a}{|r|}\right)^3 \frac{I_2 - I_1}{I_3} \sin(2(\varphi - \lambda))$$



We compute the one–period map of the libration angle:

$$(\varphi(0), \dot{\varphi}(0)) \longmapsto (\varphi(T), \dot{\varphi}(T))$$

where T is the period of the Keplerian orbit.

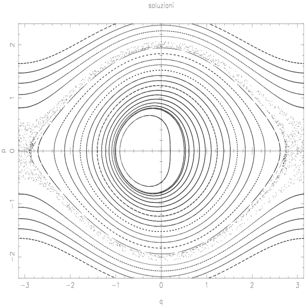


Example: resonances of quasi-integrable systems (periodically perturbed pendulum)

- The differential equation is:

$$\ddot{\varphi} = -\sin(\varphi) + \varepsilon f(\varphi, t) \qquad (2)$$

with f which is 2π periodic.



- Example:
$\ddot{\varphi} = -\sin(\varphi) + \varepsilon\cos(\varphi)\cos(t)$,
$\varepsilon = 0.02$

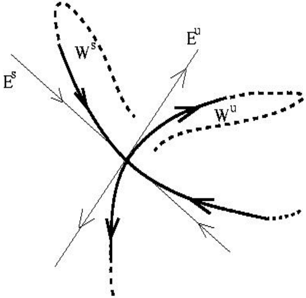
Stable and unstable manifolds of saddle points

- In all these system there is a saddle fixed point z for any value of the parameters (ε or e)
- We define the stable manifold of z:

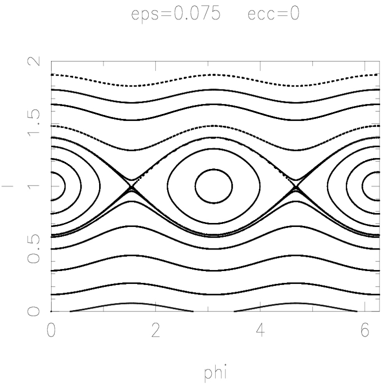
$$W^s = \{x : \lim_{t\to\infty} \psi^t(x) = z\}$$

and the unstable manifold of z:

$$W^u = \{x : \lim_{t\to\infty} \psi^{-t}(x) = z\}$$

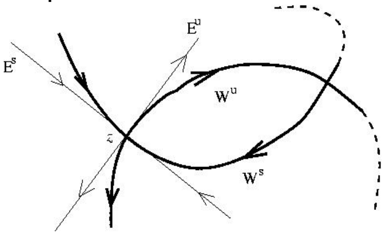


- Stable manifold theorem: W^s, W^u are differential manifolds, locally tangent in the fixed point to the egenvectors of $D\psi(z)$ (linear stable and unstable spaces).



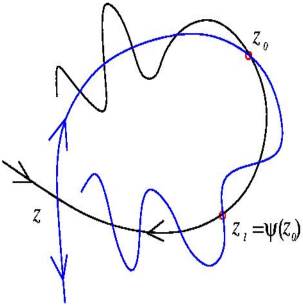
- In 2D systems with a first integral (pendulum, rotations of satellites with $e = 0$) the manifolds W^s,W^u are contained in the level sets of a regular function.

To produce chaos we need an additional hypothesis:



- We suppose that there exists at least an homoclinic point of intersection of W^s, W^u.

The existence of an homoclinic point is sufficient to show that W^s, W^u have a complicate topology

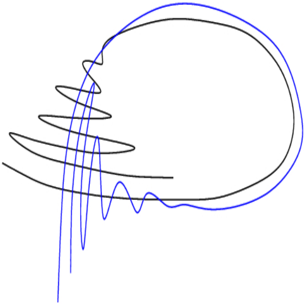


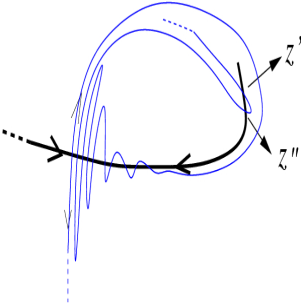
- Each point of the orbit of the homoclinic point z_0:

$$z_t = \psi^t(z_0) \quad t \in \mathbb{Z}$$

  is homoclinic (it is on $W^s \cap W^u$ and the intersection is transverse).

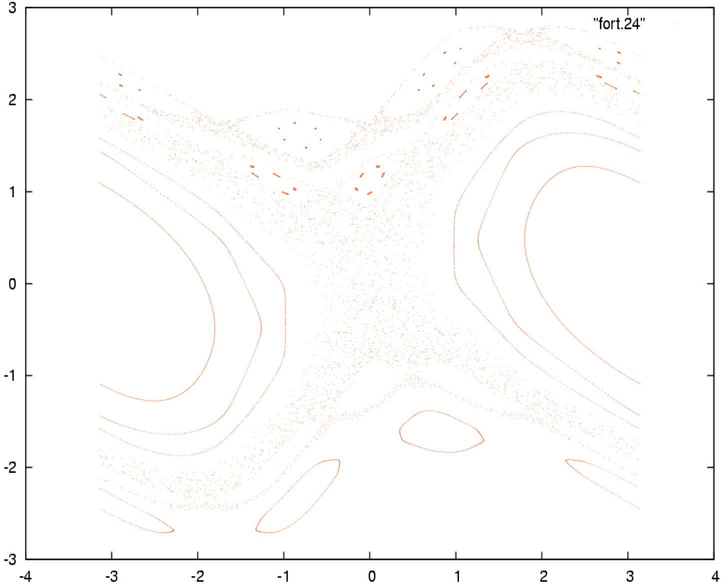
- The unstable manifold cuts the stable manifold transversally infinite times

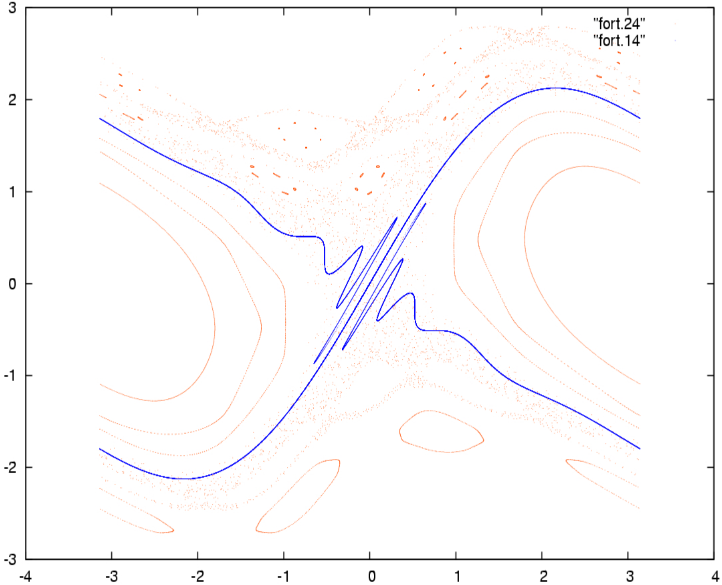




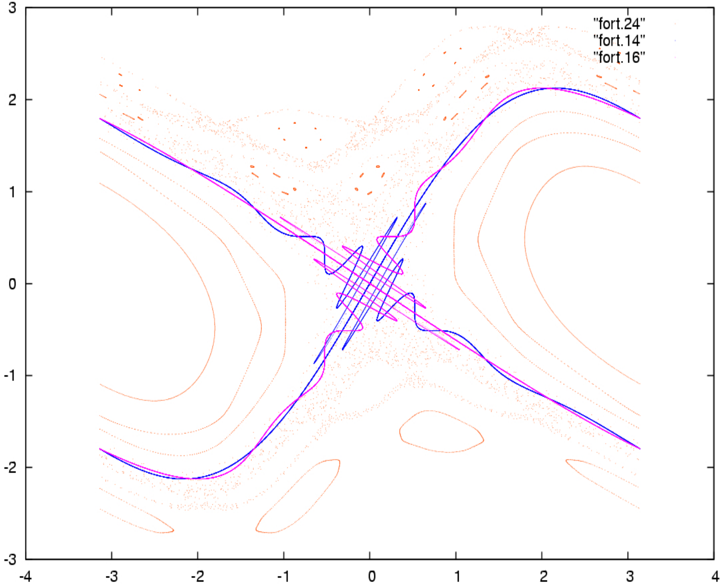
- Approaching the fixed point the base of each lobe becomes smaller and the height becomes bigger (near the fixed point there is contraction along E^s and expansion along E^u).

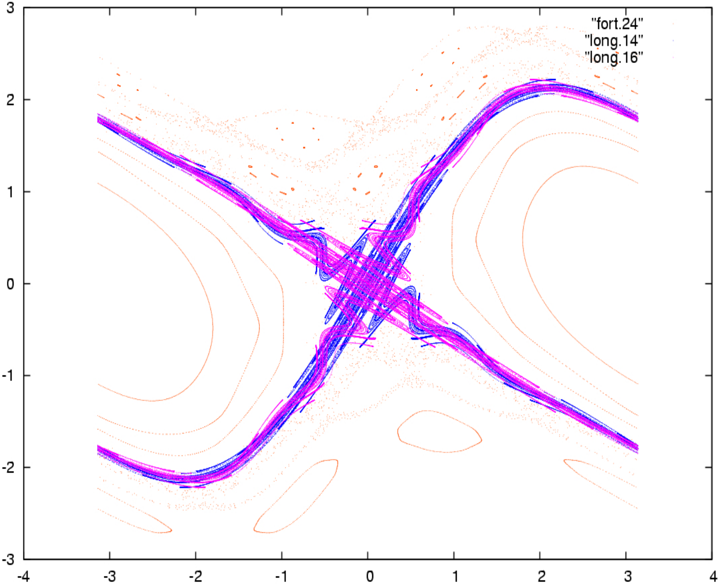
- At a given point, the lobes of the unstable manifold are so big that they are forced to intersect the stable manifold in points z', z'' which are not in the orbit of z_0. The orbits of these points contains only homoclinic points.....


"fort.24"
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4



"fort.24"
"fort.14"
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4


"fort.24"
"fort.14"
"fort.16"
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4


"fort.24"
"long.14"
"long.16"
3
2
1
0
-1
-2
-3
-4
-3
-2
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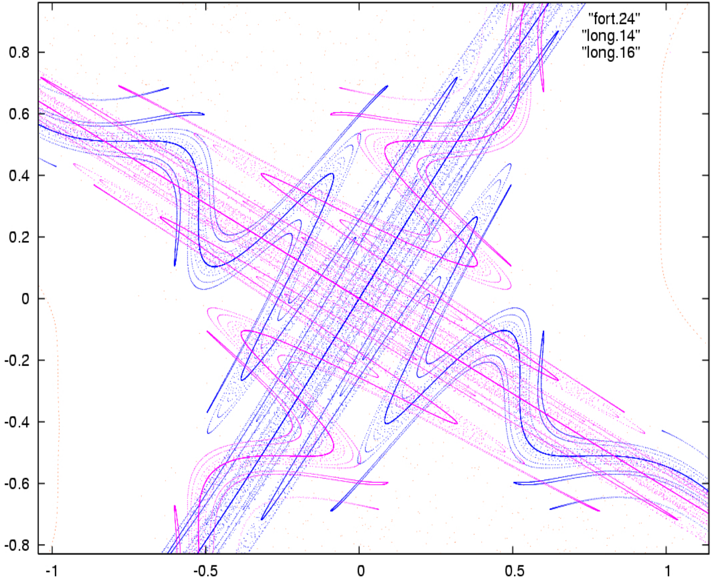
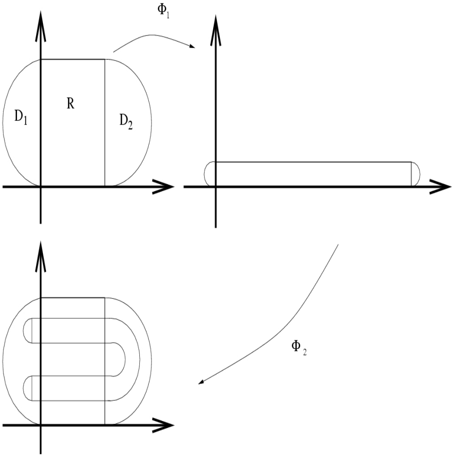
0
1
2
3
4


"fort.24"
"long.14"
"long.16"
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1

The horseshoe map



- The dynamics in the homoclinic tangles can be described using symbolic dynamics and horseshoe conjugation.

- The orbits of points in D_1 or D_2 converge to a fixed point of D_1
- The orbits of points $z \in R$ such that after some iterations of the map belong to D_1 or D_2 converge to z
- We are interested only in the points whose orbit belongs to R:

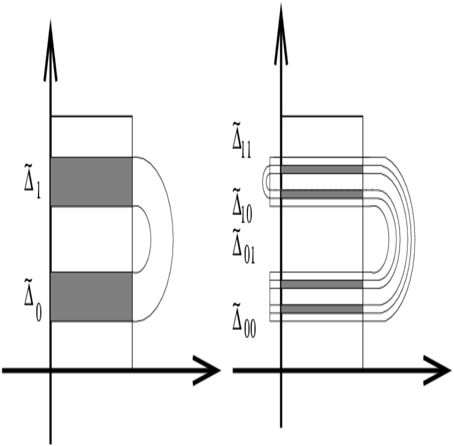
$$\Lambda_+ = \cap_{k \geq 0} \phi^{-k}(R)$$

and are image of points of R:

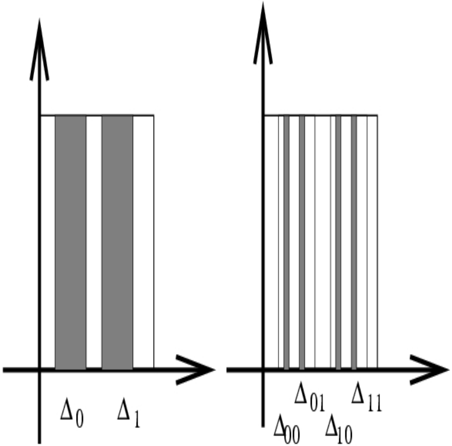
$$\Lambda_- = \cap_{k \geq 0} \phi^{k}(R)$$

so that:

$$\Lambda = \Lambda_+ \cap \Lambda_-$$



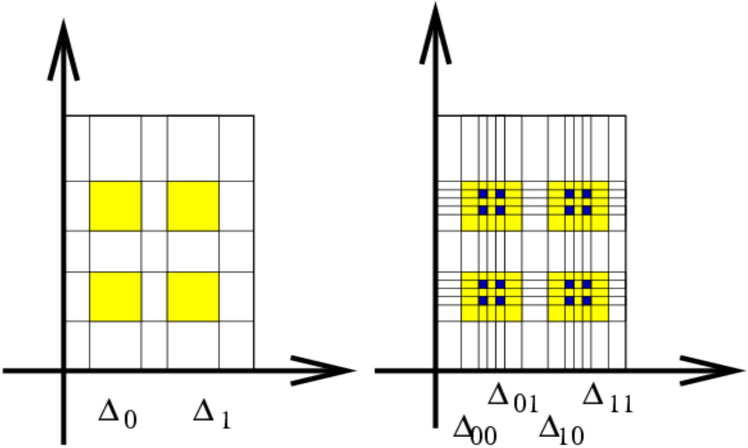
- $R \cap \phi(R)$: points of R which after 1 iteration of the map are still in R
- $R \cap \phi(R) \cap \phi^2(R)$: points of R which after 2 iterations of the map are still in R
- Λ_-: has the structure of a Cantor set of horizontal lines
- Each horizontal line can be labelled by an infinite sequence $\sigma_1, \sigma_2, \ldots$, $\sigma_k \in \{0, 1\}$



- $R \cap \phi^{-1}(R)$
- $R \cap \phi^{-1}(R) \cap \phi^{-2}(R)$
- Λ_+: has the structure of a Cantor set of vertical lines
- Each vertical line can be labelled by an infinite sequence $\sigma_1, \sigma_2, \ldots$, $\sigma_k \in \{0, 1\}$

The set Λ has the structure of a double Cantor set, each point is in the intersection of a vertical and an horizontal line and can be labelled by a double inifinite sequence

$$\sigma = (....\sigma_{-1}, \sigma_0, \sigma_1, ...) \quad , \sigma_k \in \{0, 1\}$$



Symbolic dynamics of the horseshoe

It is convenient to label the points of Λ in the following way: we define the "symbolic story" of a point $z \in \Lambda$; that is an invertible map:

$$h : \Lambda \longrightarrow \Sigma$$

$$z \longmapsto \sigma$$

where:

$$\Sigma = \{\sigma = (....\sigma_{-1}, \sigma_0, \sigma_1, ...) \ \ , \sigma_k \in \{0, 1\} \ \ \forall k\}$$

such that for any point $z \in \Lambda$ it is:

$$\phi^t(z) \in \Delta_\sigma \ \ t \in \mathbb{Z}$$

that is $\sigma_t = 0$ if $\phi^t(z) \in \Delta_0$, $\sigma_t = 1$ if $\phi^t(z) \in \Delta_1$.

- h is invertible
- h is homeomorphism ($\sigma_k = \sigma'_k$ for all $|k| \leq N$, then dist$(\sigma, \sigma') \leq 1/2^N$)
- The symbolic dynamics of ϕ is the map:

$$\tilde{\phi}(\sigma)_k = \sigma_{k+1} \quad ,$$

i.e. the left–shift of the sequence σ

For any arbitrary sequence of symbols there exists a point z with that story!!!!

- there are two fixed points: (......00000000000......), (....1111111....)
- there are periodic orbits of any period (one repeats a finite sequence $\sigma_0, \ldots, \sigma_{T-1}$)
- the set of points whose orbits are periodic is dense: near any (........01001111.........) there exist the periodic sequence (...010011110100111101001111.....).

- there exist points whose orbit is dense in Λ:

$$0\ 1\quad 00\ 01\ 11\ 10\quad 000\ 010\ 101\ 001\ 110\ 111....$$

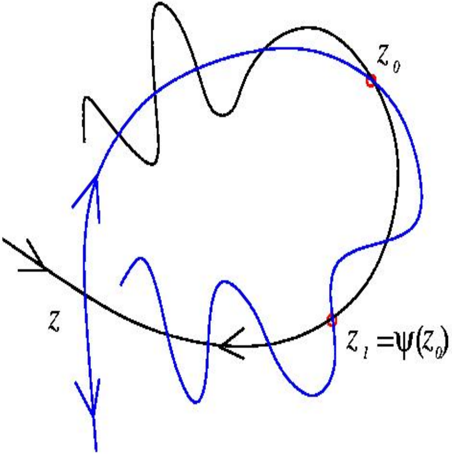
- There is exponential divergence of close initial conditions in Λ.
- Error ε in the initial condition implies that the symbolic story of z is known up to a finite time $|\log \varepsilon|$

All these facts can be used to characterize non–trivial, chaotic, dynamics on Λ.

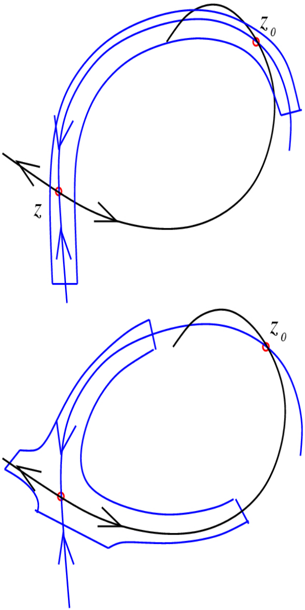
“The dynamics of the horseshoe is chaotic”

Horshoes in the homoclinic tangles

We find horseshoes (or similar maps) in the homoclinic tangles.

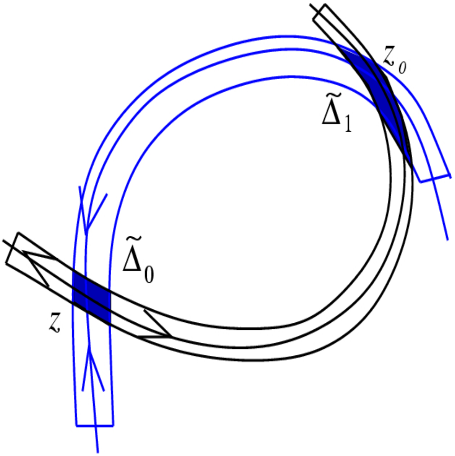


- There is an horseshoe in this system with a saddle point z and an homoclinic point

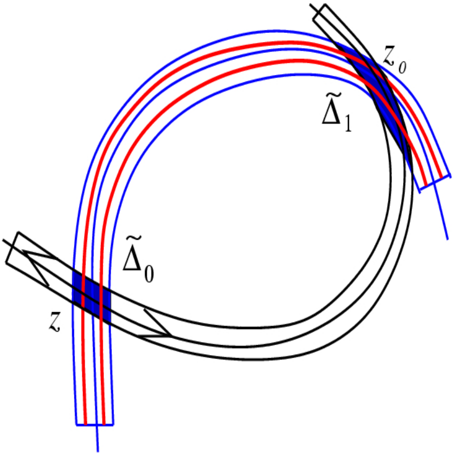


- We chose a set along the stable manifold, which contains the fixed point and an homoclinic point

- The image of this set is compressed along the stable manifold and stretched along the unstable manifold.

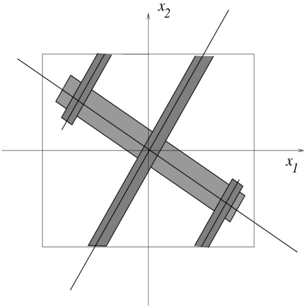


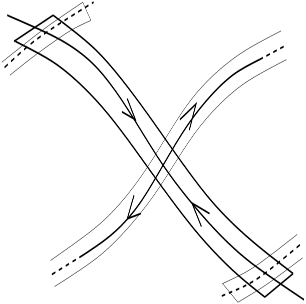
- After some iterations k the image cuts the original 'rectangle' set in two 'horizontal' sets $\tilde{\Delta}_0$, $\tilde{\Delta}_1$, whose pre–images are two 'vertical sets' Δ_0, Δ_1. We have obtained an horseshoe.



- After some iterations k the image cuts the original 'rectangle' set in two 'horizontal' sets $\tilde{\Delta}_0$, $\tilde{\Delta}_1$, whose pre–images are two 'vertical sets' Δ_0, Δ_1. We have obtained an horseshoe.

Symbolic conjugation to $\sigma_k \in \{0, -1, 1\}$





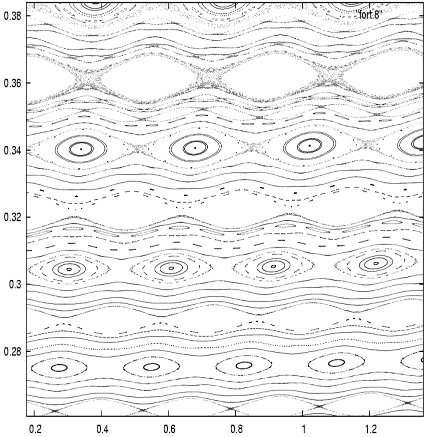
- Symbolic dynamics in the Arnold cat.
- Symbolic dynamics in the forced pendulum, spin-orbit librations.

In conclusion.....

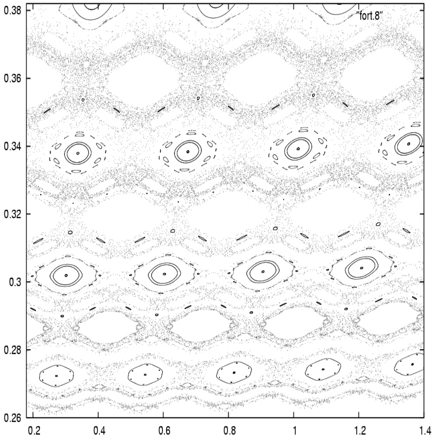
- Non-trivial, chaotic dynamics are produced near the homoclinic tangles of hyperbolic fixed points
- The existence of an homoclinic point is a minimum requirement for the existence of a complicate structure for the stable and unstable manifolds of the fixed point
- It is possible to identify horseshoes (or similar maps) for some iterates of the dynamical systems: we can describe the dynamics by means of the symbolic dynamics.
- Existence of dense sets of periodic orbits, of dense orbits, of arbitrary 'stories', exponential divergence of close initial conditions, short prediction time.

Barriers to chaotic diffusion

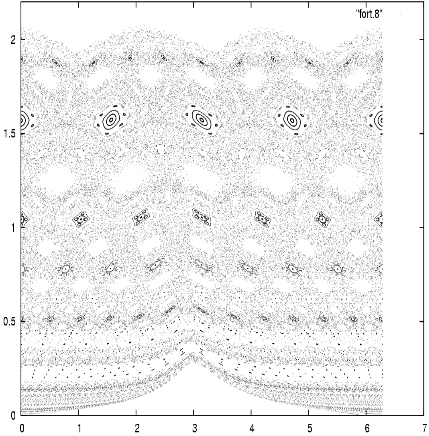
In 2D systems invariant curves are topological barriers to diffusion.



- $\varphi_2' = \varphi_2 + I_2$
  $I_2' = I_2 + \varepsilon \frac{\sin \varphi_2'}{(\cos \varphi_2' + 1.1)^2}$,
  $\varepsilon = 0.002$
- Chaotic motions are bounded by invariant curves: the action I is confined near the initial value for infinite times.
- KAM theorem: for small ε there is a big volume of invariant curves



- $\varepsilon = 2.6\ 10^{-3}$
- limit situation: most of the invariant barriers to the diffusion of variable I seem to have disappeared



- $\varepsilon = 4\ 10^{-3}$
- All the invariant barriers to the diffusion of variable *I* have disappeared

Invariant sets with given rotation number α

We consider the map:

$$\varphi' = \varphi + I$$

$$I' = I + \varepsilon f(\varphi')$$

and we look for invariant sets M with given rotation number by looking for parameterizations of $\mathbb{S}^1$ which conjugate the dynamics to a fixed rotation of a number α:

$$\mathbb{S}^1 \to \mathbb{R} \times \mathbb{S}^1$$

$$\vartheta \mapsto (I = V(\vartheta), \varphi = U(\vartheta))$$

such that the dynamics in the variable ϑ is:

$$\vartheta \longmapsto \vartheta + \alpha \quad .$$

Therefore, replacing $I = V(\vartheta)$, $\varphi = U(\vartheta)$, $\varphi' = U(\vartheta + \alpha)$, $I' = V(\vartheta + \alpha)$ in the equations defining the map, one obtains:

$$\varphi' = \varphi + I \qquad \Rightarrow \qquad V(\vartheta) = U^+ - U$$

$$I' = I + \varepsilon f(\varphi') \qquad \Rightarrow \qquad U - \frac{U^+ + U^-}{2} + \frac{\varepsilon}{2} f \circ U = 0$$

where $U^+ = U(\vartheta + \alpha)$, $U^- = U(\vartheta - \alpha)$.

Theorem 1 (Mather-Aubry) For any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ there exists a solution U to equations:

$$U - \frac{U^+ + U^-}{2} + \frac{\varepsilon}{2} f \circ U = 0 \quad ,$$

which is strictly monotone and 'periodic':
$U(\vartheta + 2\pi) = U(\vartheta) + 2\pi$.

Theorem 2 (KAM) If $\alpha/(2\pi) \in \mathbb{R}\backslash\mathbb{Q}$ is Diophantine, i.e.:

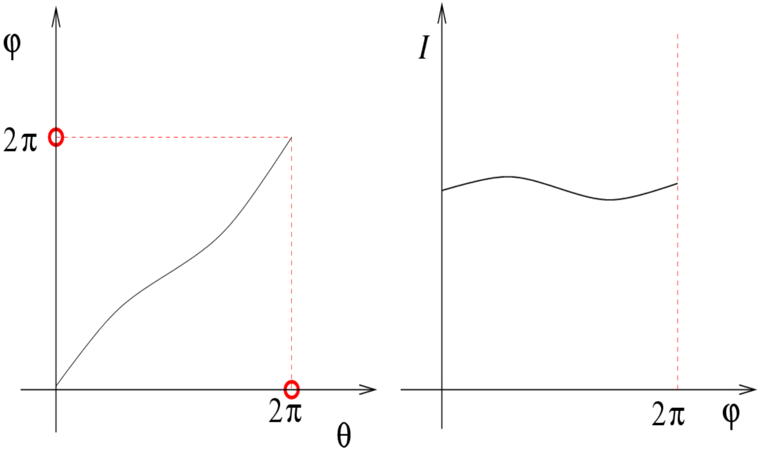
$$\left| \frac{\alpha}{2\pi} - \frac{p}{q} \right| \geq \frac{\gamma}{q^\tau} \ \ \forall p \in \mathbb{Z} \ \ q \in \mathbb{N}$$

and ε is small, then U is smooth.

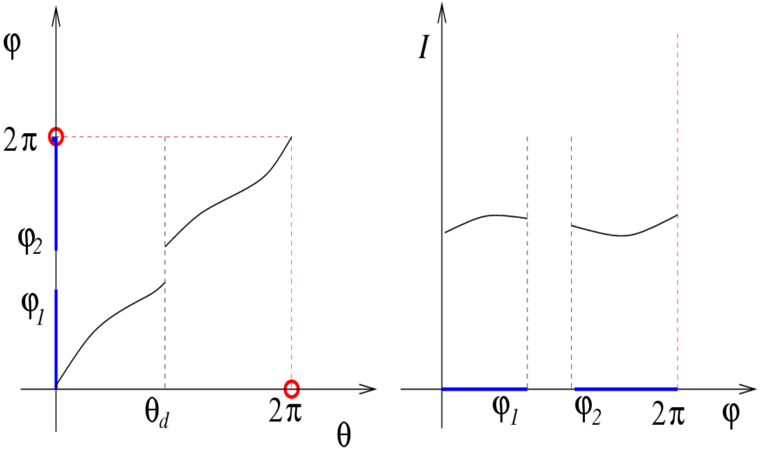
Given a rotation number α, a solution U, the set:

$$M_\alpha = \{(V(\vartheta) = U^+(\vartheta) - U(\vartheta), U(\vartheta))\}$$

is invariant set of motions with rotation number α.
If U is continuous, M_α is a curve: KAM curve, complete barrier to diffusion.

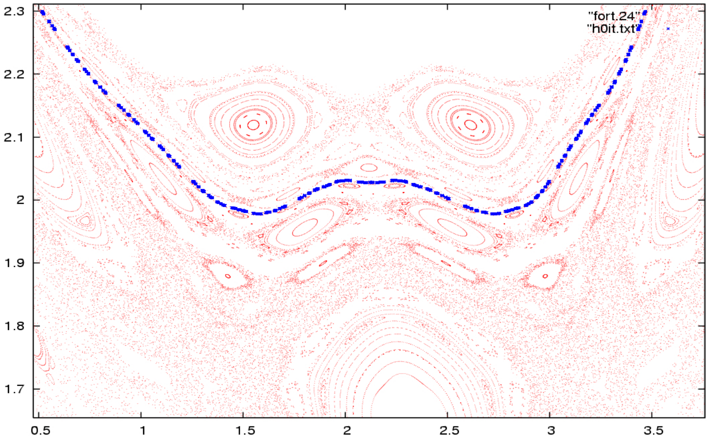


For any discontinuity of U the set M_α has a 'hole' in the phase space.



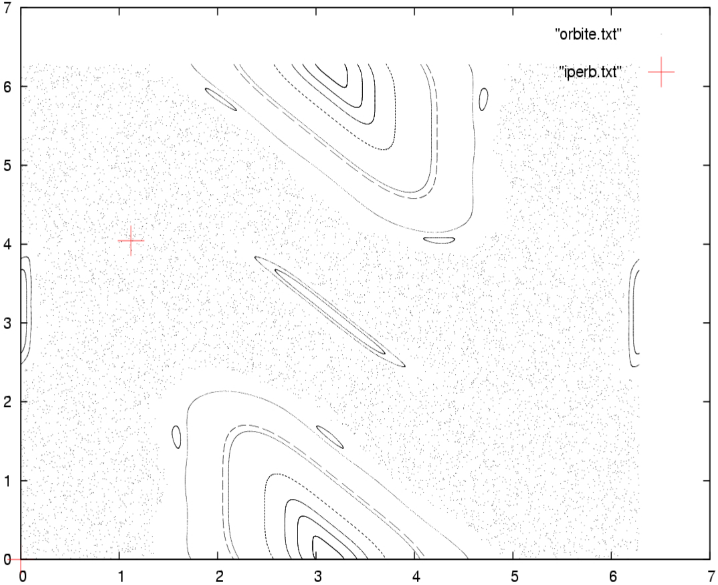
There is at most a countable set of discontinuities: M_α is called cantorus, partial barrier to diffusion.

Standard map, $\varepsilon \sim 0.98$

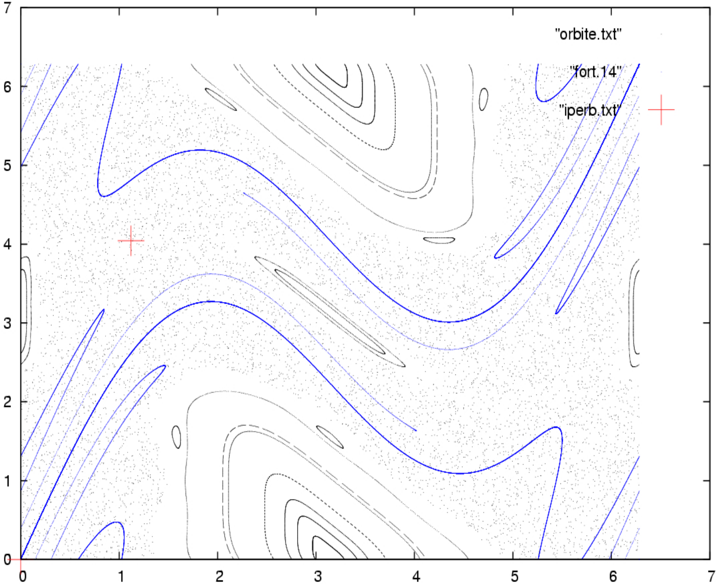


Diffusion along eteroclinic points

- By computing the fixed saddle points of ψ^k: one gets the hyperbolic periodic orbits of period k
- Compute their stable and unstable manifolds
- If the stable manifold of an hyperbolic periodic orbit intersects transversally the unstable manifold of a periodic hyperbolic orbit there is the possibility of passing from the chaotic region related to one periodic orbit to the other: Chirikov diffusion.



"orbite.txt"
"iperb.txt"
0
1
2
3
4
5
6
7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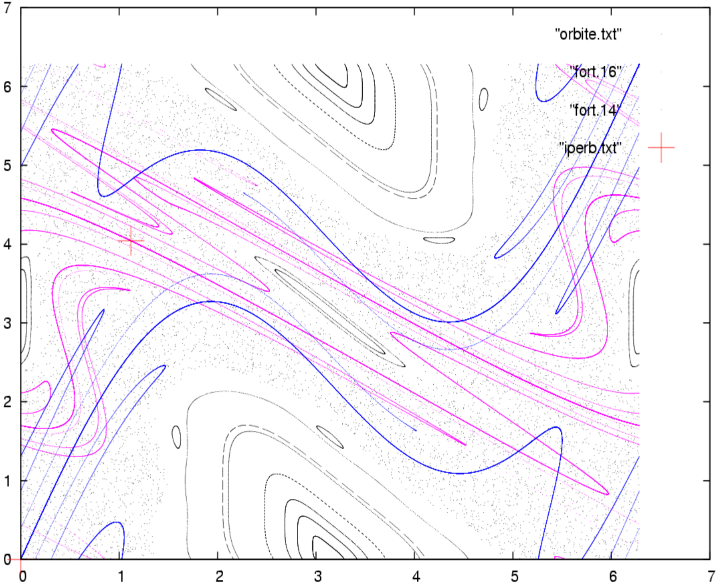


"orbite.txt"
"fort.16"
"fort.14"
"iperb.txt"
0
1
2
3
4
5
6
7

In conclusion.....

- In 2D systems invariant curves are complete barriers to diffusion
- KAM theorem: if ε is small, there are many KAM curves which prevent global diffusion; only local chaos is allowed
- Increasing ε the KAM curves break into cantori
- Cantori are partial barriers to diffusion, but are effective only for ε near the critical value of breaking of the KAM curve
- Etheroclinic connections give the possibility of Chirikov diffusion

The celebrate model of Arnold diffusion

A special quasi–integrable system:

$$H = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon \cos\varphi_1 + \varepsilon\mu(\cos\varphi_1 - 1)(\sin\varphi_2 + \sin t)$$

with Hamilton equations:

$$\begin{aligned} \dot\varphi_1 &= I_1 \\ \dot\varphi_2 &= I_2 \\ \dot I_1 &= \varepsilon \sin\varphi_1 + \varepsilon\mu \sin\varphi_1(\sin\varphi_2 + \sin t) \\ \dot I_2 &= -\varepsilon\mu(\cos\varphi_1 - 1)\cos\varphi_2 \end{aligned} \tag{3}$$

For $\varepsilon = 0$ the system has only 3D invariant tori:

$$\dot\varphi_1 = I_1 = \ const \qquad \dot\varphi_2 = I_2 = \ const \qquad \dot t = 1$$

For $\varepsilon \neq 0$ we consider the special resonance:

$$\dot{\varphi}_1 = 0$$

which contains the manifold of invariant 2D tori:

$$I_1 = 0 \qquad \varphi_1 = 0 \qquad \dot{\varphi}_2 = I_2(0) \quad \dot{t} = 1$$

which are hyperbolic:

- For $\mu = 0$ the system is the product of a pendulum and a rotator
$$H = \frac{I_1^2}{2} + \varepsilon \cos \varphi_1 + \frac{I_2^2}{2}$$
The stable/unstable manifolds of each torus are the separatrices of the pendulum: there is not diffusion of I_2
- For $\mu \neq 0$ the unstable manifold of each hyperbolic torus intersects transversally the stable manifolds of close invariant tori: there exists diffusion through these etheroclinic points.

This kind of diffusion is called Arnold diffusion

- Recent generalization to systems

$$H = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon \cos \varphi_1 + \varepsilon \mu f(\varphi_1, \varphi_2, t)$$

- For generic quasi–integrable systems:

$$H = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon f(\varphi_1, \varphi_2, t)$$

there is not a proof of existence of Arnold diffusion

Regular and chaotic motions in 4D systems

Two coupled twist maps as model example:

$$\varphi_1' = \varphi_1 + I_1 \quad , \quad \varphi_2' = \varphi_2 + I_2$$

$$I_1' = I_1 - \varepsilon \frac{\partial f}{\partial \varphi_1}(\varphi_1', \varphi_2') \quad , \quad I_2' = I_2 - \varepsilon \frac{\partial f}{\partial \varphi_2}(\varphi_1', \varphi_2')$$

where ε is 'small' and:

$$f = \frac{1}{\cos(\varphi_1) + \cos(\varphi_2) + 4}$$

- If $\varepsilon = 0$ the actions are constants of motion and the angles rotate at constant angular velocity
- If ε is small there is a big volume of 2D invariant tori (KAM theorem)
- The 2D invariant tori do not trap motions in the 4D phase space: there is the possibility of diffusion even for very small ε
- Diffusion needs hyperbolic structures

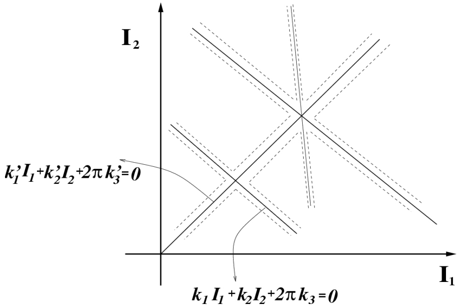
Resonances of the system

The angle $k_1\varphi_1 + k_2\varphi_2$ is resonant when:

$$k_1\varphi_1' + k_2\varphi_2' = (k_1\varphi_1 + k_2\varphi_2) + (k_1 I_1 + k_2 I_2) = (k_1\varphi_1 + k_2\varphi_2) + 2\pi k_3$$

i.e. when:

$$k_1 I_1 + k_2 I_2 - 2\pi k_3 = 0 \quad .$$



Far from all these resonances we find the KAM tori:

$$|k_1 I_1 + k_2 I_2 - 2\pi k_3| \geq \frac{\mathcal{O}(\sqrt{\varepsilon})}{|(k_1, k_2, k_3)|^\tau}$$

- KAM tori fill a set of volume $1 - \sqrt{\varepsilon}$
- The complement of the set of invariant tori is called Arnold web, and contains the possible hyperbolic structures, chaotic motions, diffusion,...

A section of the–phase space

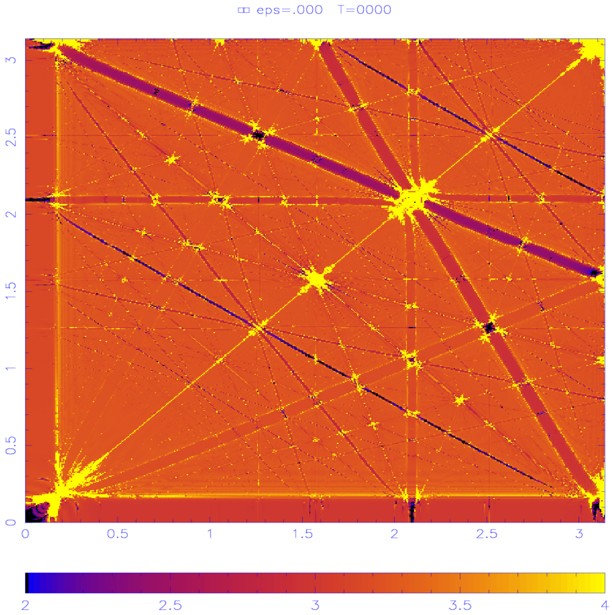
To represent the Arnold web and the hyperbolic structures we consider a 2D section of the phase space which is transverse to all KAM tori:

$$S = \{(I_1, I_2, \varphi_1, \varphi_2) : \ \ (\varphi_1, \varphi_2) = (0, 0) \ \}$$

For each point of the section I compute the Fast Lyapunov Indicator, and represent it with a color scale:

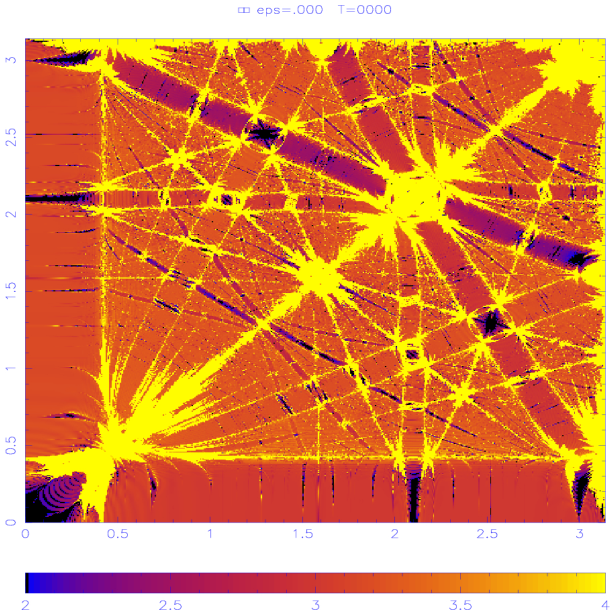
- higher values (yellow) $\rightarrow$ chaotic resonant motions
- intermediate values (orange) $\rightarrow$ regular motions (KAM tori)
- lower values **(black)** $\rightarrow$ **regular resonant librations**

Computation of the Arnold web



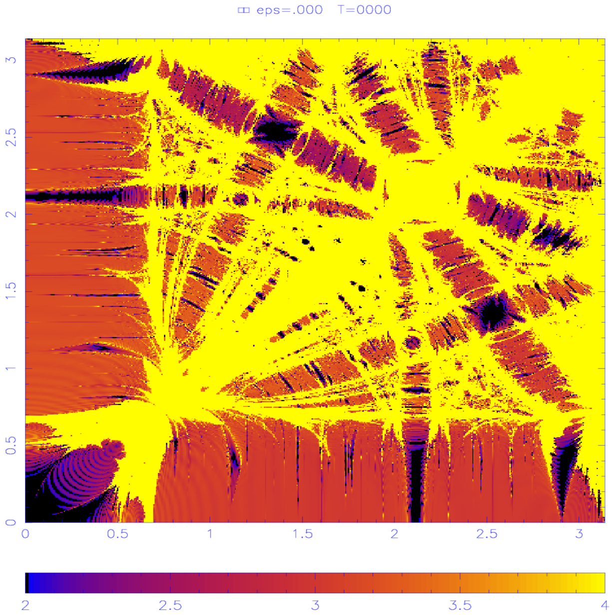
- Surface: $\varphi_1 = 0$, $\varphi_2 = 0$
- $\varepsilon = 0.1$: there is a prevalence of KAM tori in the phase space
- The hyperbolic structures are organized in the web of resonances

Computation of the Arnold web



- Surface: $\varphi_1 = 0$, $\varphi_2 = 0$
- $\varepsilon = 0.6$: there is still a prevalence of KAM tori in the phase space,
- Resonances are enlarged

Computation of the Arnold web



- Surface: $\varphi_1 = 0$, $\varphi_2 = 0$
- $\varepsilon = 1.6$: there is a prevalence of resonant motions: resonant regular motions and chaotic motions
- The hyperbolic structures are not organized in a web

Two possible scenarios for diffusion:

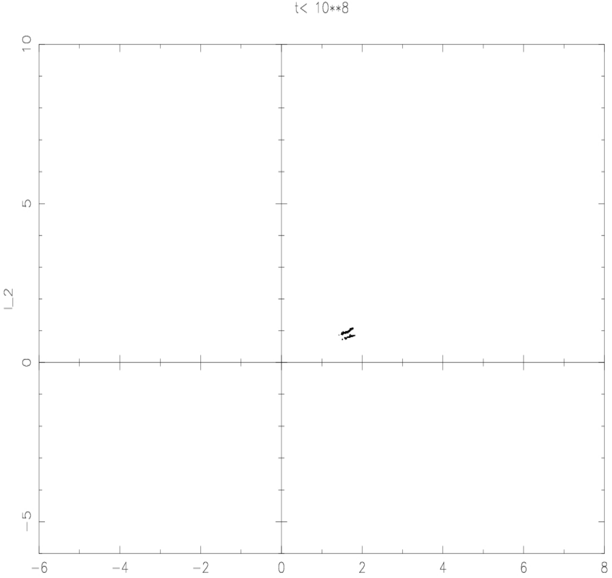
- If ε is small
  - diffusion is confined to the Arnold web, which is open, dense set of small measure.
  - By Nekhoroshev theorem it is extremely slow:

$$|I(t) - I(0)| \leq \mathcal{O}(\varepsilon^\alpha) \ , \ \ |t| \leq \exp -\left(\frac{\varepsilon_0}{\varepsilon}\right)^\beta$$

  - If this kind of diffusion exists, it could be called Arnold diffusion.
- If ε is big
  - resonances overlap globally, diffusion is possible in a big volume of the phase–space
  - this kind of diffusion is fast, and could be called Chirikov diffusion.

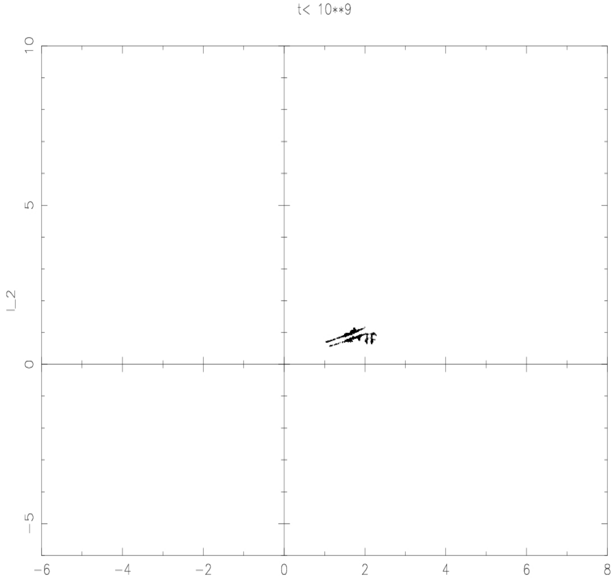
Diffusion of 20 orbits

For $\varepsilon = 0.6$ the orbits diffuse through the Arnold web of the system



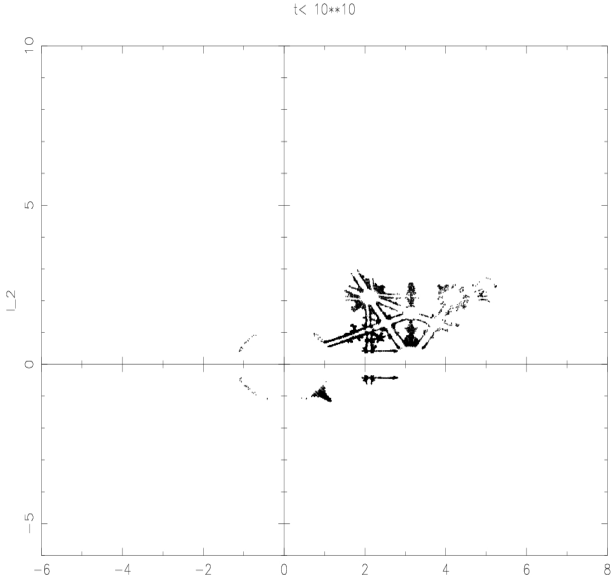
Diffusion of 20 orbits

For $\varepsilon = 0.6$ the orbits diffuse through the Arnold web of the system



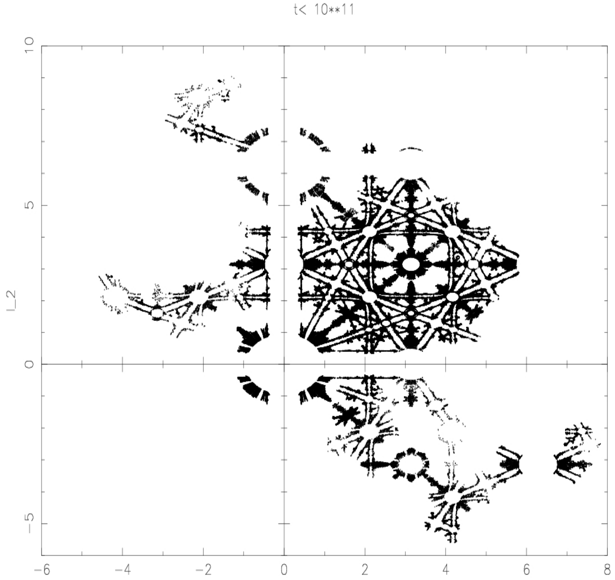
Diffusion of 20 orbits

For $\varepsilon = 0.6$ the orbits diffuse through the Arnold web of the system



Diffusion of 20 orbits

For $\varepsilon = 0.6$ the orbits diffuse through the Arnold web of the system



- Can we apply these techniques to planetary systems?
- The mean motion resonances among 3 of more planets are arranged as a a web similar to the Arnold web
- Applications: outer planets of our Solar System; extra-solar planetary systems; asteroids of the main belt.

The outer Solar System is near (but not in) some important mean–motion resonances, such as:

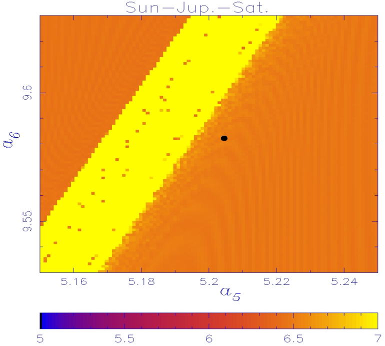
- $2\dot{\lambda}_J - 5\dot{\lambda}_S \sim 0$ (Great Inequality)
- $\dot{\lambda}_S - 3\dot{\lambda}_U \sim 0$
- $\dot{\lambda}_U - 2\dot{\lambda}_N \sim 0$

These quasi–resonances are coupled through 2 order planetary masses producing three–planet resonances

Example: the restricted Sun–Jupiter–Saturn system

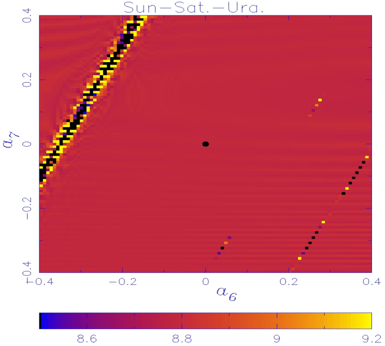
- The Sun–Jupiter–Saturn system is near the $2\dot{\lambda}_J - 5\dot{\lambda}_S$.
- We compute the Fast Lyapunov Indicator for a grid of initial conditions obtained by fixing all initial conditions except for a_5, a_6.
- FLI color scale:
  - higher values (yellow) $\rightarrow$ chaotic resonant motions
  - intermediate values (orange) $\rightarrow$ regular motions (KAM tori)
  - lower values **(black)** $\rightarrow$ **regular resonant motions**

Geometry of resonances for the Sun–Jupiter–Saturn three body problem:



- The 5–2 resonance appears as a large yellow band.
- The SJS system is near it, but not in it
- Outside the resonance, motions are regular

Geometry of resonances for the Sun–Saturn–Uranus three body problem:



- The 3–1 resonance appears as a large yellow band.
- The SSU system is near it, but not in it
- Outside the resonance, motions are regular

Because the system is outside two–planets mean motion resonances, one can average the hamiltonian at first order order in planetary masses obtaining:

$$H = h(L_5, ..., L_8) + \varepsilon\, H_0(L, G, H, g, h) + \varepsilon^2\, H_1(L, G, H, l, g, h; \varepsilon) \quad ,$$

where H_1 contains harmonics:

$$n_i\lambda_i + n_j\lambda_j + n_k\lambda_k + \sum_h (k_h\omega_h + k'_h\Omega_h)$$

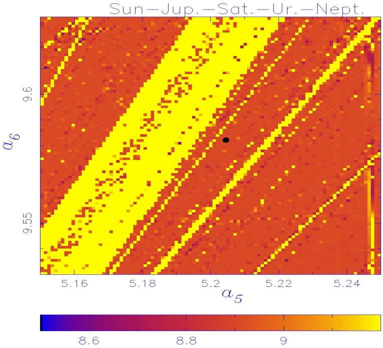
Diagonalization/normalization of the secular part H_0 (far from secular resonances):

$$(L, G, H, l, g, h) \quad \longmapsto \quad (L', G,' H', l', g', h')$$

such that (dropping primes):

$$\mathcal{H} = h(L_5, ..., L_8) + \varepsilon\, \mathcal{H}_0(L, G, H) + \varepsilon^2\, \mathcal{H}_1(L, G, H, l, g, h; \varepsilon)$$

The terms $\mathcal{O}(\varepsilon^2)$ are important for the dynamics:



The integration of the Sun–Jupiter–Saturn–Uranus–Neptune gives a very different result: many 3 planet resonances have appeared

What are three–planet resonances

- Linear combinations of the quasi–resonances:

$$(2\dot{\lambda}_J - 5\dot{\lambda}_S)\ ,\ (\dot{\lambda}_S - 3\dot{\lambda}_U)\ ,\ (\dot{\lambda}_U - 2\dot{\lambda}_N)$$

produce combinations such that:

$$n_i\dot{\lambda}_i + n_j\dot{\lambda}_j + n_k\dot{\lambda}_k \sim \text{secular frequencies}$$

- As a consequence, some angles:

$$n_i\lambda_i + n_j\lambda_j + n_k\lambda_k + \sum_h (k_h\omega_h + k'_h\Omega_h)$$

are in resonance.

For each n_i, n_j, n_k such that:

$$n_i\dot{\lambda}_i + n_j\dot{\lambda}_j + n_k\dot{\lambda}_k \sim \text{secular frequencies}$$

there is a cluster/multiplet of resonances:

$$n_i\dot{\lambda}_i + n_j\dot{\lambda}_j + n_k\dot{\lambda}_k + \sum_h (k_h\dot{\omega}_h + k'_h\dot{\Omega}_h) \sim 0$$

for different values of the secular coefficients k_h, K'_h.

Representation in the a_i, a_j plane of the cluster:

$$n_i \dot{\lambda}_i + n_j \dot{\lambda}_j + n_k \dot{\lambda}_k + \sum_h (k_h \dot{\omega}_h + k'_h \dot{\Omega}_h) \sim 0$$

is a family of parallel resonances at distance $\mathcal{O}(\varepsilon)$, and amplitude $\mathcal{O}(\varepsilon)$: there can be resonance overlapping.

Near the initial condition of Jupiter–Saturn–Uranus–Neptune: amplitude/distance of these resonances is of order $10^{-4}/10^{-5}$ AU.

High resolution spans of three–planet resonances

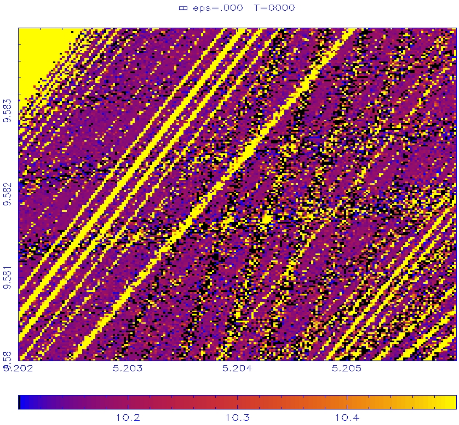
We need to zoom near initial condition of the true solar system: typical amplitude and separation of these resonances is of $\Delta a \sim 10^{-6}/10^{-4} AU$.

- What we compute: we fix all orbital elements, except for (two) semi–major axis, which span a two dimensional grid around the true initial condition. For each point of the grid we compute the Fast Lyapunov Indicator.
- What we expect to find: any fixed integers n_i, n_j, n_k generate a family of parallel resonances: in the semi–major axis space

$$n_i \dot{\lambda}_i + n_j \dot{\lambda}_j + n_k \dot{\lambda}_k + \sum_h (k_h \dot{\omega}_h + k'_h \dot{\Omega}_h) \sim 0$$

Jupiter and Saturn

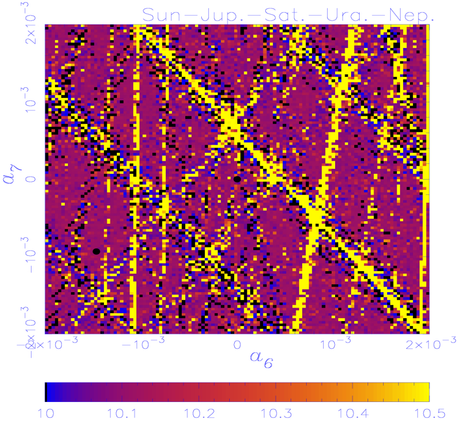
All initial conditions are fixed except for a_5, a_6



- Many families of 3PR at the distance of 10^{-5} AU
- The 3PR are arranged as a regular web (similar to the Arnold web)
- The true solar system is very near a crossing of resonances: possibility of chaotic motions.

Saturn and Uranus

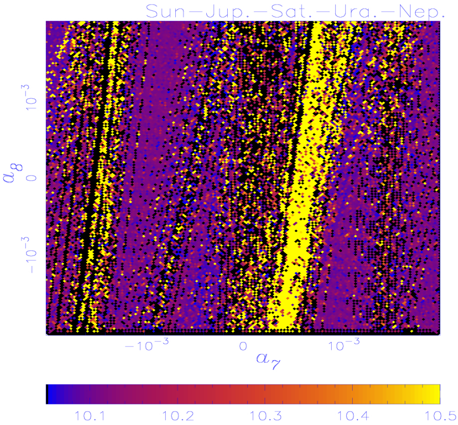
All initial conditions are fixed except for a_6, a_7



- Many families of 3PR at the distance of 10^{-5} AU
- The 3PR are arranged as a regular web (similar to the Arnold web)
- The true solar system is very near a crossing of resonances: possibility of chaotic motions.

Uranus and Neptune

All initial conditions are fixed except for a_7, a_8



- There is local overlapping of resonances around the true initial condition.
- The chaotic region extends by 10^{-3} AU around the true initial conditions.
- Possibility of chaotic Chirikov–like diffusion.

Families of TPR responsible of the overlapping

Families of resonances:

$$n_J \dot{\lambda}_J + n_S \dot{\lambda}_S + n_U \dot{\lambda}_U + n_N \dot{\lambda}_N \sim 0$$

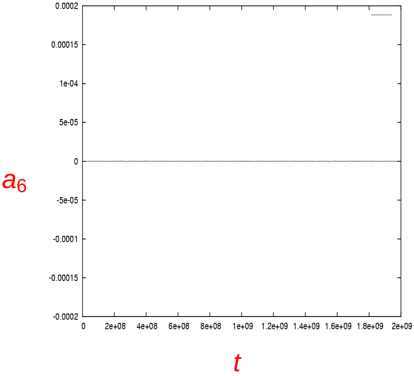
responsible of the overlapping:

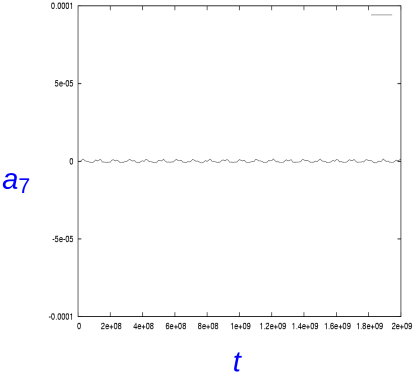
- $(n_J, n_S, n_U, n_N) = (1, -3, 3, -3)$,
- $(n_J, n_S, n_U, n_N) = (3, -5, -7, 0)$ (Murray-Holman)
- Their integer combinations, such as $(n_J, n_S, n_U, n_N) = (2, -2, -10, 3)$.

Dynamics outside TPR

The dynamics is quasi–periodic (KAM–tori like).

Evolution of a_6 and a_7, averaged over 10 Myr.





Dynamics inside one isolated family of TPR

Dynamics inside the family:

$$n_i \dot{\lambda}_i + n_j \dot{\lambda}_j + n_k \dot{\lambda}_k + ... \sim 0$$

where only n_i, n_j are fixed.

- Motions can be chaotic with $T_L \sim 10 Myr$;
- The (averaged) motion of the actions is mainly flattened along the line parallel to

$$(\dot{L}_i, \dot{L}_j, \dot{L}_k) \sim (n_i, n_j, n_k)$$

- Diffusion of the actions in other directions occur in much longer time (Arnold diffusion–like), typically:

$$\frac{\Delta a}{\Delta t} \sim 10^{-6} \frac{AU}{Gyr}$$

Dynamics inside overlapping TPR

Dynamics inside the overlapping families:

$$n_i\dot{\lambda}_i + n_j\dot{\lambda}_j + n_k\dot{\lambda}_k + ... \sim 0 \qquad n'_i\dot{\lambda}_i + n'_j\dot{\lambda}_j + n'_k\dot{\lambda}_k + ... \sim 0$$

with the first one larger than the second one.

- Chaos: $T_L \sim 10 Myr$;
- (averaged) actions is flattened on:

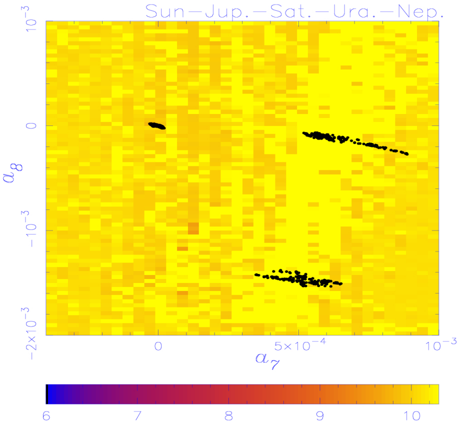
$$(\dot{L}_i, \dot{L}_j, \dot{L}_k) \sim (n_i, n_j, n_k)$$

but the second family forces chaotic diffusion on a 2-dimensional space; with:

$$\frac{\Delta a}{\Delta t} \sim 10^{-4}\,\frac{AU}{Gyr}$$

Dynamics in the TPR

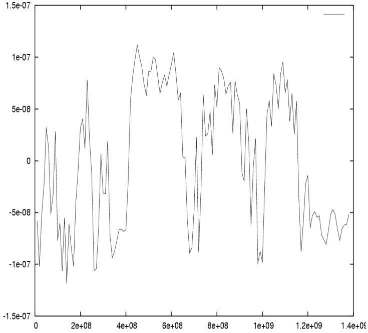
Dynamics in the TPR



- Projection of the dynamics (averaged over 20 Myr) on the a_7, a_8 space

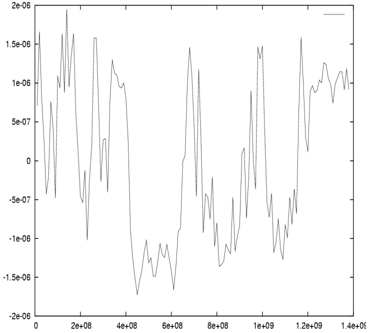
Dynamics of the true Solar System

Evolution of a_5 and a_6, averaged over 10 Myr



a_5

t



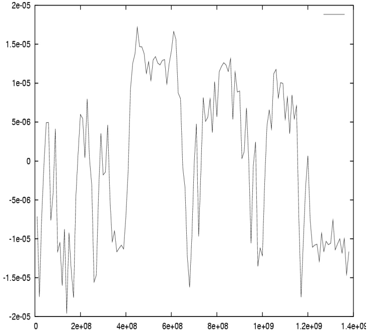
a_6

t

Dynamics of the true Solar System

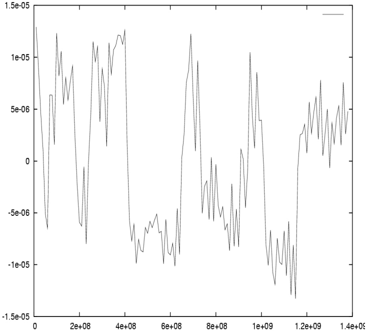
Evolution of a_7 and a_8, averaged over 10 Myr

a_7



t

a_8



t

Conclusions

- The crossing of the resonances related to $(n_1, n_2, n_3, n_4) = (1, -3, 3, -3), (3, -5, -7, 0)$ (and their integer combinations) is responsible of a region of overlapping resonances which extends of about 10^{-3} AU with respect to a_7, a_8.
- Fictitious planetary systems with initial conditions in this region can undergo very slow systematic drifts of a_7, a_8.
- Our solar system is marginally in this region: the semi–major axes a_7, a_8 can undergo systematic variations of $\sim 10^{-4}$ AU/Gyr.
- The dynamical state of the giant planets is metastable: on 10^7 yrs they seem trapped in thin three–planet resonances responsible of the chaos, but on 10^9 yrs these resonances cause a drift of the orbital elements.