ASYMPTOTIC SERIES FOR PLANETARY MOTION IN PERIODIC TERMS IN THREE DIMENSIONS

P. J. MESSAGE

Department of Applied Mathematics and Theoretical Physics, Liverpool University, England

Abstract. For the 'planetary case' of the gravitational n-body problem in three dimensions, a sequence of Lie series contact transformations is used to construct asymptotic series representations for the canonical parameters of the instantaneous orbits in a Jacobi formulation. The series contain only periodic terms, the frequencies being linear combinations of those of the planetary orbits and those of the secular variations of the apses and nodes, and the series are in powers of the masses of the planets in terms of that of the primary, and of a quantity of the order of the excursions of the eccentricities and inclinations of the orbits. The treatment avoids singularities for circular and coplanar orbits. It follows that the major axes are given by series of periodic terms only, to all orders in the planetary masses.

1. Introduction

It is a very old question in celestial mechanics, whether the motions within the planetary system, if they were subject only to the mutual gravitational attractions of its members, would remain indefinitely of their present general nature, that is to say with the planets moving in nearly circular and nearly coplanar orbits of about their present sizes, or whether the mutual attractions alone would lead to the ultimate disruption of these features. Most, and until very recently all, of the attempts to approach an answer to this question have necessarily made use of methods of perturbation theory, which lead to expressions for the perturbations of the orbital elements in terms of multiple Fourier series. Now such expressions have been known since the time of Poincaré (1893, chapter XIII) not to be uniformly convergent, but, if appropriately derived, asymptotic in the appropriate perturbation parameter, and so do not yield firm conclusions about the properties of the motions for time intervals of unlimited length. Nevertheless, expressions of such type have been proved, by comparison both with observations of planetary positions over several centuries, and also with numerical integrations representing time intervals of the order of a million years, to be very many more times more accurate over long time intervals than we are entitled to conclude on the basis of any error bounds for such expressions so far derived by strict mathematical analysis. So we still await general methods by which the true reliability of such expressions may be analytically derived.

Laplace and Lagrange showed that, to first order in the perturbations, the major semi-axes of the planetary orbits are subject only to the sum of periodic perturbations, and their theory of the 'secular variations' indicated stability, showing that, if short-period terms are omitted from the equations for the perturbations, as well as terms of

* Proceedings of the Conference on 'Analytical Methods and Ephemerides: Theory and Observations of the Moon and Planets'. Facultés universitaires Notre Dame de la Paix, Namur, Belgium, 28–31 July, 1980.

Celestial Mechanics **26** (1982) 25–39. 0008–8714/82/0261–0025 $02.25.

second and higher degree in the eccentricities and inclinations of the orbits, then the long-term perturbations in those quantities took the form of superposed periodic oscillations, of periods between 50000 and 2000000 years (Lagrange 1781). There remains the effects of terms of higher order in the eccentricities and inclinations, and also those of higher order in the perturbing masses, arising from the interaction of short-periodic terms of lower order. For a discussion of the researches concerning secular terms in the major semi-axes, of second or higher order, see Duriez (1978). It is a result of the developments described in this paper, that such terms represent the beginning of power series expansions of periodic terms of the very long periods encountered in the theory of the secular variations.

Newcomb (1874) described a method of successive approximations to lead to expressions for the rectangular co-ordinates of the planets, consisting only of periodic terms, some of very long period. Poincaré (1893, chapter VIII) subsequently introduced the concept of asymptotic series, and also a method of contact transformation (called by him 'Lindstedt's method', and more recently called 'von Zeipel's', but surely really due to Poincaré himself) by use of which it can be assured that each successive approximation satisfies the equations of motion to higher order in the small parameter of approximation than its predecessor, and that no secular terms appear at higher order. He applied this method, with various other devices, to the general three-body gravitational problem, to show how it yields expressions for the orbital elements consisting only of periodic terms. The Lie series method of contact transformation (Hori, 1966) has the additional advantage of giving explicit expressions for the new variables in terms of the old, and vice-versa, and so avoids any potential difficulties at higher order associated with the inversion of implicit expressions. In an earlier presentation (Message, 1976) this type of transformation was used, with some modifications of Poincaré's approach, to show in principle how to produce explicit expressions, in asymptotic series, with periodic terms only, for the osculating elliptic elements in the n-body gravitational problem in the plane in the planetary case, that is, where one of the bodies (the 'primary') is much more massive than the others, and where the orbits of the others about the primary are nearly circular. In the present paper, the extension of this treatment is made to three dimensions, on the supposition that the orbits are nearly coplanar, and some other changes are made. Some of the transformations are carried out in a different sequence to the corresponding ones in Poincaré's treatment. First are introduced the rectangular-type orbital elements to represent the orientations and eccentricities of the orbits, before the short-period terms are eliminated, to avoid any suggestion of possible singularities corresponding to zero values of the orbital eccentricities of inclinations, which of course really represent especially simple cases, and not singular ones. The short-period terms are then eliminated before the treatment of the long-period problem, so that the matrices used in that treatment are shown to be functions only of the momenta Λ_j, which are constants of motion of that problem, so that these matrices are constant. The conclusion is that, within the validity of these asymptotic expansions, the perturbations in these rectangular orbital elements, and the major semi-axes, are the superpositions of periodic terms only (some of the long periods of the secular variations). (It is to be noted, of course, that the expansion by orders of the perturbations is carried out in a different way from that used in Poisson's method, and so the terms of a particular order cannot be identified with the terms of that order obtained by Poisson's method.)

Duriez (1977, 1979) has constructed a 'global theory' of the perturbations of the planetary orbits, entirely in periodic terms, using a set of non-canonical parameters for the osculating orbits, and with the use of an electronic computer to obtain the numerical coefficients in the expressions for those parameters.

2. Formulation of the Problem

We consider, then, a system consisting of a particle P_0 (the 'primary', or 'Sun'), and n other particles $P_1, P_2, \ldots, P_n$ (the 'planets'), moving under their mutual gravitational attractions. We denote the mass of P_j by m_j, and suppose each of $m_1, m_2, \ldots, m_n$ to be small compared to m_0. We use Jacobi's system of relative position vectors, that is,

$$\mathbf{x}_1 = \overrightarrow{P_0P_1}, \text{ and, for } 2 \leqslant j \leqslant n,\ \mathbf{x}_j = \overrightarrow{G_{j-1}P_j}, \tag{1}$$

where G_j is the centre of mass of $P_0, P_1, \ldots, P_j$. It follows that

$$\overrightarrow{P_0P_j} = \mathbf{x}_j + \sum_{k=1}^{j-1} \frac{m_k \mathbf{x}_k}{M_k}$$

and

$$\overrightarrow{P_0G_j} = \sum_{k=1}^{j} \frac{m_k \mathbf{x}_k}{M_k} \quad (j = 2, 3, \ldots, n), \tag{2}$$

where

$$M_j = \sum_{k=0}^{j} m_k, \quad (j = 1, 2, \ldots, n). \tag{3}$$

Let us put $\varepsilon = \max_{1 \leqslant j \leqslant n}(m_j/m_0)$, which gives a measure of the ratio of the largest perturbation on a planet by another planet, to the solar attraction on that planet, and let us choose quantities β_j so that

$$\varepsilon\beta_j = m_j M_{j-1}/M_j \quad (j = 2, 3, \ldots, n), \tag{4}$$

and also choose μ_j so that

$$\varepsilon\mu_j\beta_j = Gm_0m_j \quad (j = 1, 2, \ldots, n), \tag{5}$$

where G is the constant of gravitation. Then the equations of motion take the form

$$\beta_j\ddot{\mathbf{x}}_j = -\mu_j\beta_j\mathbf{x}_j/|\mathbf{x}_j|^3 + \frac{\partial R}{\partial \mathbf{x}_j} \quad (j = 1, 2, \ldots, n), \tag{6}$$

where R is a function of the quantities ε, β_j, $|\mathbf{x}_j|$, and $\mathbf{x}_j \cdot \mathbf{x}_k$ with the property that $R \to 0$ as $\varepsilon \to 0$. In this limit, the problem separates into n independent inverse square central force problems. We also note that

$$m_j = \varepsilon\beta_j/\{1 - \varepsilon\beta_j/M_{j-1}\},$$

and so that

$$\mu_j = G\{m_0 + \varepsilon\beta_j + 0(\varepsilon^2)\}.$$

If we take $\mathbf{p}_j = \beta_j\dot{\mathbf{x}}_j$ as the momentum conjugate to $\mathbf{x}_j$, then the equations of motion may be derived from the Hamiltonian function

$$H(\mathbf{x}, \mathbf{p}) = \sum_{j=1}^{n} \{\tfrac{1}{2}\mathbf{p}_j^2/\beta_j - \mu_j\beta_j/|\mathbf{x}_j|\} - R \tag{7}$$

(where $\mathbf{x}$ denotes the set of n vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, and $\mathbf{p}$ the set of n vectors $\mathbf{p}_1$, $\mathbf{p}_2, \ldots, \mathbf{p}_n$). Now each of the relative position vectors $\mathbf{x}_j$ and its conjugate momentum $\mathbf{p}_j$ at any time defines an instantaneous Keplerian orbit in the central force problem corresponding to $\varepsilon = 0$, and so we may describe the state of relative motion of the system at any time by means of the n corresponding osculating orbits. Suppose that the osculating orbit corresponding to $\mathbf{x}_j$ and $\mathbf{p}_j = \beta_j\dot{\mathbf{x}}_j$ has major semi-axis a_j, eccentricity e_j, and that, referred to a fixed plane and fixed initial direction, its orbital inclination is I_j, the longitude of its ascending node is Ω_j, the longitude of its apse is ϖ_j, and the mean longitude of P_j in the orbit is λ_j. Then a set of canonical co-ordinates for the whole system is

$$(\lambda_1, \lambda_2, \ldots, \lambda_n, \varpi_1, \varpi_2, \ldots, \varpi_n, \Omega_1, \Omega_2, \ldots, \Omega_n)$$

and their conjugate momenta are, respectively,

$$(\Lambda_1, \Lambda_2, \ldots, \Lambda_n,\ \Pi_1, \Pi_2, \ldots, \Pi_n, N_1, N_2, \ldots, N_n),$$

where

$$\Lambda_j = \beta_j\sqrt{(\mu_j a_j)},$$
$$\Pi_j = \Lambda_j\{\sqrt{(1 - e_j^2)} - 1\},$$

and

$$N_j = \Lambda_j\sqrt{(1 - e_j^2)}\cdot(\cos I_j - 1), \tag{8}$$

the Hamiltonian function being

$$H^+(\lambda, \varpi, \Omega; \Lambda, \Pi, N) = -\tfrac{1}{2}\sum_{j=1}^{n} \beta_j^3\mu_j^3/\Lambda_j^2 - R. \tag{9}$$

(where λ denotes the set $\lambda_1, \lambda_2, \ldots, \lambda_n$, ϖ denotes $\varpi_1, \varpi_2, \ldots, \varpi_n$, and likewise for Ω, Λ, π, and N, and this type of notation will be used henceforth to denote n-tuples.) We will need to note that $|\mathbf{x}_j|$ and $\mathbf{x}_j \cdot \mathbf{x}_k$, and hence all quantities constructed from these, have period 2π in each of the co-ordinate angles λ_j, ϖ_j, and Ω_j, and that $|\mathbf{x}_j|$

and $\mathbf{x}_j \cdot \mathbf{x}_k$ have the d'Alembert property, that is, their dependence on the e_j, I_j, ϖ_j, and Ω_j is such that they may be expressed in power series (with positive powers only) in the $e_j \sin \varpi_j$, $e_j \cos \varpi_j$, $\gamma_j \sin \Omega_j$, and $\gamma_j \cos \Omega_j$, (where $\gamma_j = \sin(\frac{1}{2}I_j)$), the coefficients depending only on the a_j and λ_j. Also the terms in $\mathbf{x}_j \cdot \mathbf{x}_k$ involving γ_j and γ_k are all of even degree in these. Because of the periodicity in the angles, R may be expanded as a multiple Fourier series

$$R = \sum_{\nu} K_\nu \cos N_\nu \tag{10}$$

where ν denotes the set of $3n$ integers $\nu_1, \nu_2, \ldots, \nu_n, \nu'_1, \nu'_2, \ldots, \nu'_n, \nu''_1, \nu''_2, \ldots, \nu''_n$, and

$$N_\nu = \sum_{j=1}^{n} (\nu_j \lambda_j + \nu'_j \varpi_j + \nu''_j \Omega_j), \tag{11}$$

the summation in (10) being over all sets ν with $\sum_{j=1}^{n} (\nu'_j + \nu'_j + \nu''_j) = 0$ and $\nu_1 \geqslant 0$. Because of the d'Alembert property, the coefficient K_ν is equal to the product of the n quantities $e_j^{|\nu'_j|} \gamma_j^{|\nu_j''|}$ multiplied by a power series in $e_1^2, e_2^2, \ldots, e_n^2, \gamma_1^2, \gamma_2^2, \ldots, \gamma_n^2$ the coefficients in that series being functions of the a_j, and $\sum_{j=1}^{n} \nu''_j$ is even or zero.

Now the apse longitude ϖ_j is undefined when $e_j = 0$, and the node longitude Ω_j is undefined when $I_j = 0$, and since the singularities so arising are not real features of the motion (circular orbits, and coplanar orbits, are simple cases, not singular ones) let us use the rectangular-type canonical osculating elements, as introduced by Poincaré:

$$\begin{aligned}
\xi_j &= \sqrt{(-2\Pi_j)} \cos \varpi_j, \\
\eta_j &= \sqrt{(-2\Pi_j)} \sin \varpi_j, \\
q_j &= \sqrt{(-2N_j)} \cos \Omega_j, \\
p_j &= \sqrt{(-2N_j)} \sin \Omega_j,
\end{aligned} \tag{12}$$

for each $j = 1, 2, \ldots, n$, noting that

$$(\lambda, \varpi, \Omega; \Lambda, \Pi, N) \mapsto (\lambda, \xi, q; \Lambda, \eta, p)$$

is a time independent contact transformation. These expressions may in fact be written

$$\begin{aligned}
\xi_j &= \sqrt{(\Lambda_j)}\, \varepsilon_j \cos \varpi_j, \\
\eta_j &= \sqrt{(\Lambda_j)}\, \varepsilon_j \sin \varpi_j, \\
q_j &= 2\sqrt{(\Lambda_j + \Pi_j)}\, \gamma_j \cos \Omega_j,
\end{aligned}$$

and

$$p_j = 2\sqrt{(\Lambda_j + \Pi_j)}\, \gamma_j \sin \Omega_j, \tag{13a}$$

where

$$\varepsilon_j = \sqrt{2} \cdot \sqrt{\{1 - \sqrt{(1 - e_j^2)}\}}. \tag{13b}$$

Hence the coefficient K_ν may be expressed as the product of the n quantities $\varepsilon_j^{|\nu_j'|}$ $\gamma_j^{|\nu_j''|}$, multiplies by a power series in the ε_j^2 and the γ_j^2, the coefficients being functions of the Λ_j. A typical term in the expansion of R is of the form

$$A_{\nu,r,s}\prod_{j=1}^{n}(\varepsilon_j^{|\nu'_j|+2r_j}\gamma_j^{|\nu''_j|+2s_j})\cos\sum_{j=1}^{n}(\nu_j\lambda_j+\nu'_j\varpi_j+\nu''_j\Omega_j), \tag{14}$$

where $A_{\nu,r,s}$ is a function of the Λ_j, and r_j and s_j are non-negative integers, and $\prod_{j=1}^{n}$ here denotes the product over all n values of j. In terms of the rectangular-type elements, the typical term becomes the real part of

$$A'_{\nu,r,s}\prod_{j=1}^{n}\{(\xi_j^2+\eta_j^2)^{r_j}(q_j^2+p_j^2)^{s_j}\}\exp\left(i\sum_{j=1}^{n}\nu_j\lambda_j\right)\times$$
$$\times\prod_{j=1}^{n}\{(\xi_j+i\sigma'_j\eta_j)^{|\nu'_j|}(q_j+i\sigma''_jp_j)^{|\nu''_j|}\}, \tag{15}$$

where $i^2=-1$, $A'_{\nu,r,s}$ is a function of the Λ_j, and σ'_j is $+1$ if $\nu'_j\geqslant 0$, and -1 otherwise, and σ''_j is $+1$ if $\nu''_j\geqslant 0$, and -1 otherwise. Thus this term is of the form

$$B_{\nu,r,s}\cos\left(\sum_{j=1}^{n}\nu_j\lambda_j\right)+C_{\nu,r,s}\sin\left(\sum_{j=1}^{n}\nu_j\lambda_j\right) \tag{16}$$

where $B_{\nu,r,s}$ and $C_{\nu,r,s}$ are polynomials in the ξ_j, η_j, q_j, and the p_j, homogeneous and of degree $|\nu'_j|+2r_j$ in ξ_j and η_j and homogeneous and of degree $|\nu''_j|+2s_j$ in q_j and p_j.

Also $B_{\nu,r,s}$ is of even degree in all the η_j and also in all the p_j, while $C'_{\nu,r,s}$ is of odd degree in all the η_j and also in all the p_j.

3. Transformation to the Long-Period Problem

To separate out the long-period problem, we use the Lie series contact transformation

$$(\lambda,\xi,q;\Lambda,\eta,p)\mapsto(\lambda',\xi',q';\Lambda',\eta',p')$$

defined by

$$\lambda_j=\lambda'_j+\frac{\partial S}{\partial\Lambda'_j}+\sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_S^p\left(\frac{\partial S}{\partial\Lambda'_j}\right),$$

$$\xi_j=\xi'_j+\frac{\partial S}{\partial\eta'_j}+\sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_S^p\left(\frac{\partial S}{\partial\eta'_j}\right),$$

$$q_j=q'_j+\frac{\partial S}{\partial p'_j}+\sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_S^p\left(\frac{\partial S}{\partial p'_j}\right),$$

$$\Lambda_j=\Lambda'_j-\frac{\partial S}{\partial\lambda'_j}-\sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_S^p\left(\frac{\partial S}{\partial\lambda'_j}\right),$$

$$\eta_j=\eta'_j-\frac{\partial S}{\partial\xi'_j}-\sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_S^p\left(\frac{\partial S}{\partial\xi'_j}\right),$$

and

$$p_j = p'_j - \frac{\partial S}{\partial q'_j} - \sum_{p=1}^{\infty} \frac{1}{(p+1)!} \mathscr{L}_S^p\left(\frac{\partial S}{\partial q'_j}\right), \tag{17}$$

where S is a function of the λ'_j, ξ'_j, q'_j, Λ'_j, η'_j, and p'_j still to be chosen, and $\mathscr{L}_S$ is the operation defined by

$$\begin{aligned}\mathscr{L}_S(f) &= \{f, S\}\\ &= \sum_{j=1}^{n} \left\{\frac{\partial f}{\partial \lambda'_j}\frac{\partial S}{\partial \Lambda'_j} + \frac{\partial f}{\partial \xi'_j}\frac{\partial S}{\partial \eta'_j} + \frac{\partial f}{\partial q'_j}\frac{\partial S}{\partial p'_j} - \right.\\ &\qquad \left. - \frac{\partial f}{\partial \Lambda'_j}\frac{\partial S}{\partial \lambda'_j} - \frac{\partial f}{\partial \eta'_j}\frac{\partial S}{\partial \xi'_j} - \frac{\partial f}{\partial p'_j}\frac{\partial S}{\partial q'_j}\right\}\end{aligned} \tag{18}$$

that is, the Poisson bracket of f and S. Also we have put

$$\mathscr{L}_S^p(f) = \mathscr{L}_S(\mathscr{L}_S^{p-1}(f)) \quad \text{for} \quad p = 2, 3, \ldots,$$

where $\mathscr{L}_S^1 \equiv \mathscr{L}_S$. If f is any function of the co-ordinates and the momenta, then we have

$$f(\lambda, \xi, q; \Lambda, \eta, p) = f(\lambda', \xi', q'; \Lambda', \eta', p') + \sum_{p=1}^{\infty} \frac{1}{p!}\mathscr{L}_S^p(f). \tag{19}$$

If $\tilde{H}(\lambda, \xi, q; \Lambda, \eta, p)$ is the Hamiltonian function for the system before this transformation, and $H'(\lambda', \xi', q'; \Lambda', \eta', p')$ is the Hamiltonian function after transformation, then since the transformation is time-independent,

$$\begin{aligned}H'(\lambda', \xi', q'; \Lambda', \eta', p') &\equiv \tilde{H}(\lambda, \xi, q; \Lambda, \eta, p)\\ &= \tilde{H}(\lambda', \xi', q'; \Lambda', \eta', p') + \sum_{p=1}^{\infty} \frac{1}{p!}\mathscr{L}_S^p(\tilde{H}),\end{aligned} \tag{20}$$

on using (19). We proceed to choose S so that H' is independent of the λ'_j, by seeking formal expansions

$$S = \sum_{p=1}^{\infty} S_p \quad \text{and} \quad H' = \sum_{p=1}^{\infty} H'_p, \tag{21}$$

where S_p and H'_p will be chosen to be of degree p in ε explicitly, that is, without having regard to the implicit dependence on ε due to the fact that μ_j, and therefore Λ_j, depend on ε as given by Equations (4) and (5). (In fact, if we denote ε by ε_1 when it appears in μ_j as expressed in terms of ε and the β_j, then S_p and H'_p are those parts of S and H' respectively of degree p in ε, not taking account of the occurrence of ε_1.) Now

$$\begin{aligned}\tilde{H} &= -\tfrac{1}{2}\sum_{j=1}^{n} \mu_j^2\beta_j^3/\Lambda_j^2 - R\\ &= \tilde{H}_0 - R,\end{aligned} \tag{22}$$

say, and we may expand

$$R = \sum_{p=1}^{\infty} R_p, \tag{23}$$

where R_p is of explicit degree p in ε, in the same manner. Then equating terms without explicit dependence on ε in Equation (20) gives

$$H'_0(\Lambda') = \tilde{H}_0(\Lambda') \tag{24}$$

and equating terms of first degree in ε explicitly in Equation (20) gives

$$\begin{aligned} H'_1 &= -R_1 + \{\tilde{H}_0, S_1\} \\ &= -R_1 - \sum_{j=1}^{n} n'_j \frac{\partial S_1}{\partial \lambda'_j}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} n'_j &= \frac{\partial \tilde{H}_0}{\partial \Lambda'_j} \\ &= \mu_j^2 \beta_j^3 / \Lambda_j'^3 \\ &= n_j(\Lambda'_j), \end{aligned}$$

say,

$$n_j(\Lambda_j) = \mu_j^2 \beta_j^3 / \Lambda_j^3$$

being the Keplerian expression for the mean motion of P_j. Then corresponding to a term of the form

$$B_{\nu,r,s} \cos\left(\sum_{j=1}^{n} \nu_j \lambda_j\right)$$

in R_1, we include in S_1 the term

$$-B_{\nu,r,s} \sin\left(\sum_{j=1}^{n} \nu_j \lambda'_j\right) \Big/ \left(\sum_{j=1}^{n} \nu_j n'_j\right),$$

and, corresponding to a term of the form

$$C_{\nu,r,s} \sin\left(\sum_{j=1}^{n} \nu_j \lambda_j\right)$$

in R_1, we include in S_1 the term

$$C_{\nu,r,s} \cos\left(\sum_j \nu_j \lambda'_j\right) \Big/ \left(\sum_j \nu_j n'_j\right),$$

so that H'_1 is left equal to the terms of $-R_1$ which are independent of the λ'_j (with ξ'_j substituted for ξ_j, q'_j for q_j, & ce.) Equating terms of second-order explicitly in ε in Equation (20) then gives

$$H'_2 = -R_2 + \{\tilde{H}_0, S_2\} + \tfrac{1}{2}\{\{\tilde{H}_0, S_1\}, S_1\}$$
$$= -R_2^* - \sum_{j=1}^{n} n'_j \frac{\partial S_2}{\partial \lambda'_j}, \tag{26}$$

where

$$R_2^* = R_2 - \tfrac{1}{2}\{\{\tilde{H}_0, S_1\}, S_1\}$$

which is known once S_1 is. (In forming these Poisson brackets, we remember that each n'_j is a function of Λ'_j.) Then S_2 may be chosen with reference to R_2^* in the same way that S_1 was chosen with reference to R_1, to ensure that H'_2 is left free of dependence on the λ'_j. In the same manner, as we proceed to each order in turn, we may choose S_p so as to leave the corresponding H'_p independent of the λ'_j, so that altogether the new Hamiltonian function H' is independent of the λ'_j, which are therefore ignorable co-ordinates in this 'secular' or 'long-period' problem in which therefore the Λ'_j are constants of the motion. (Note that this process requires there to be no linear relations between the n'_j of the sort $\sum_{j=1}^{n} \nu_j n'_j = 0$ for integers ν_j. For discussion of conditions for convergence or divergence of series of the type obtained for S_p, see Poincaré (1893, Section 147), Arnol'd (1963).)

4. Treatment of the Secular Problem

We see then that H' is composed of terms, each of the type given in the expression (8), but independent of the λ'_j, that is, which each of the $\nu_j = 0$, so that each is the real part of an expression of the form

$$A'_{\nu,r,s} \prod_{j=1}^{n} \{(\xi_j'^2 + \eta_j'^2)^{r_j} (q_j'^2 + p_j'^2)^{s_j} (\xi'_j + i\sigma'_j \eta'_j)^{|\nu'_j|} (q'_j + i\sigma'_j p'_j)^{|\nu''_j|}\} \tag{27}$$

with $\sum_{j=1}^{n} (\nu'_j + \nu''_j) = 0$, and such that $\sum_{j=1}^{n} \nu''_j$ is an even integer. Thus H' is a function of the Λ'_j and quantities of the types $\xi_j'^2 + \eta_j'^2$, $\xi'_j\xi'_k + \eta'_j\eta'_k$, $q_j'^2 + p_j'^2$, $q'_jq'_k + p'_jp'_k$, and $\xi'_jq'_j + \eta'_jp'_j$. The terms of lowest degree in the ξ'_j, q'_j, η'_j, and p'_j (apart from the term independent of them) are of degree two, and necessarily take the form

$$\sum_{(j,k)} \{K_{jk}(\xi'_j\xi'_k + \eta'_j\eta'_k) + K^*_{jk}(q'_jq'_k + p'_jp'_k)\} \tag{28}$$

the summation being over all pairs (j, k), the coefficients K_{jk} and K^*_{jk} being functions of the Λ'_j only (Clearly we may always choose these coefficients so that $K_{jk} = K_{kj}$ and $K^*_{jk} = K^*_{kj}$.) The 'secular variations' theory of Laplace and Lagrange corresponds to the retention in H'_1 of terms of this type only, so that the equations of motion for the rectangular-type variables ξ'_j, q'_j, η'_j, and p'_j in that theory are linear in those variables. We now set out to construct a theory in which terms of all orders are retained. Linear transformations may be chosen to diagonalise the quadratic forms in (28), defining

$$\sigma_r = \sum_{j=1}^{n} a_{rj}\xi'_j, \quad \tau_r = \sum_{j=1}^{n} a_{rj}\eta'_j,$$

$$\sigma_{n+r} = \sum_{j=1}^{n} a_{n+r,n+j}q'_j, \quad \tau_{n+r} = \sum_{j=1}^{n} a_{n+r,n+j}p'_j, \quad \text{for } r = 1, 2, \ldots, n, \tag{29}$$

where the a_{rj} are functions of the Λ'_j, to be chosen. In order to ensure that

$$(\lambda', \xi', q'; \Lambda', \eta', p') \mapsto (\lambda'', \sigma; \Lambda'', \tau)$$

(where σ denotes the $2n$-tuple $\sigma_1, \sigma_2, \ldots, \sigma_{2n}$, and similarly for τ) is a contact transformation, we find that we must have

$$\text{(i)} \quad \sum_{r=1}^{n} a_{rj}a_{rk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$$\text{(ii)} \quad \sum_{r=1}^{n} a_{n+r,n+j}a_{n+r,n+k} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$$\text{(iii)} \quad \Lambda'_j = \Lambda''_j \quad \text{for} \quad j = 1, 2, \ldots, n,$$

and

$$\text{(iv)} \quad \lambda'_j = \lambda''_j - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{k=1}^{n} \tau_r \frac{\partial a_{rk}}{\partial \Lambda'_j} a_{sk}\sigma_s, \quad \text{for} \quad j = 1, 2, \ldots, n, \tag{30}$$

so that, if we choose $a_{n+r,j} = a_{r,n+j} = 0$ for $1 \leqslant r \leqslant n$ and $1 \leqslant j \leqslant n$, then the $2n$ by $2n$ matrix (a_{rj}) is orthogonal. Hence the equations (29) may be inverted to give

$$\xi'_j = \sum_{r=1}^{n} a_{rj}\sigma_r, \quad \eta'_j = \sum_{r=1}^{n} a_{rj}\tau_r,$$

$$q'_j = \sum_{r=1}^{n} a_{n+r,n+j}\sigma_{n+r}, \quad p'_j = \sum_{r=1}^{n} a_{n+r,n+j}\tau_{n+r}, \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{31}$$

The transformation is time-independent, so the new Hamiltonian function, H'', is given by

$$H''(\lambda'', \sigma; \Lambda'', \tau) \equiv H'(\lambda', \xi', q'; \Lambda', \eta', p').$$

The quantities $\xi_j'^2 + \eta_j'^2$, $\xi'_j\xi'_k + \eta'_j\eta'_k$, $q_j'^2 + p_j'^2$, $q'_jq'_k + p'_jp'_k$, and $q'_j\xi'_k + p'_j\eta'_k$ may be expressed in terms of quantities of the type $\sigma_r^2 + \tau_r^2$ and $\sigma_r\sigma_s + \tau_r\tau_s$, and so H'' may be expressed in terms of these latter quantities and of the Λ''_j. We may choose the a_{rs} so that

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{rj}K_{jk}a_{sk} = \begin{cases} 0 & \text{if} \quad r \neq s \\ \frac{1}{2}A_r & \text{if} \quad r = s \quad (\text{for } r = 1, 2, \ldots n) \end{cases}$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{n+r,n+j} K^*_{jk} a_{n+s,n+k} = \begin{cases} 0 & \text{if} \quad r \neq s \\ \frac{1}{2}A_{n+r} & \text{if} \quad r = s \text{ (for } r = 1, 2, \ldots, n) \end{cases} \tag{32}$$

if the eigenvalues, A_r, and A_{n+r}, of the matrices (K_{jk}) and (K^*_{jk}), respectively, are all different. (These eigenvalues are, to first order in the ε, the frequencies in the Laplace and Lagrange secular variation theory.) Then

$$H'' = -\frac{1}{2}\sum_{j=1}^{n} \mu_j^2 \beta_j^3 / \Lambda_j''^2 - K_0 + \frac{1}{2}\sum_{r=1}^{2n} A_r(\sigma_r^2 + \tau_r^2)$$
$$+ \text{ terms of higher order in the } \sigma_r \text{ and the } \tau_r, \tag{33}$$

where K_0 is a function of the Λ_j'' only, of order ε.

We now return the problem to action and angle form by making the further contact transformation

$$(\lambda'', \sigma; \Lambda'', \tau) \mapsto (\lambda''', \omega; \Lambda''', \Omega)$$

defined by

$$\sigma_r = \sqrt{(-2\Omega_r)}\cos\omega_r$$
$$\tau_r = \sqrt{(-2\Omega_r)}\sin\omega_r \quad (r = 1, 2, \ldots, 2n)$$

and

$$\lambda_j''' = \lambda_j'', \Lambda_j''' = \Lambda_j'' \quad (j = 1, 2, \ldots, n), \tag{34}$$

The new Hamiltonian function H''' is given by

$$H'''(\lambda''', \omega; \Lambda''', \Omega) \equiv H''(\lambda'', \sigma; \Lambda'', \tau).$$

From (34) we see that $\sigma_r\sigma_s + \tau_r\tau_s = 2\sqrt{(\Omega_r\Omega_s)}\cos(\omega_r - \omega_s)$, so that H''' is expressible in terms of quantities of this sort and of the Ω_r and the Λ_j''', and so may be expanded as a multiple Fourier series of the type

$$H''' = \sum_{\nu} K_\nu(\Lambda''', \Omega)\cdot\cos\left(\sum_{r=1}^{2n} \nu_r\omega_r\right),$$

the coefficient K_ν having $2n$ factors each of the form $\Omega_r^{1/2|\nu_r|}$, and the summation is over all $2n$-tuples $\nu = (\nu_1, \nu_2, \ldots\ldots, \nu_{2n})$ for which $\sum_{r=1}^{2n} \nu_r = 0$. Then

$$H''' = -\frac{1}{2}\sum_{j=1}^{n} \mu_j^2\beta_j^3/\Lambda_j'''^2 - K_0 - R''', \tag{35a}$$

where

$$R''' = -\sum_{r=1}^{2n} A_r\Omega_r + \text{terms of higher order in the } \Omega_r. \tag{35b}$$

The Ω_r are quantities of the order of the square of the eccentricities of the orbits, and of their inclinations to the reference plane, and we suppose all of these to be small, or to be able to be chosen small, in the planetary problem. We can proceed by use of a Lie series type transformation to derive series expressions to represent the complete secular motions, if we use a small parameter of the small size of the Ω_r. In fact, denote by ρ a constant chosen to be of the order of magnitude of the largest of the Ω_r, and the expressions will be developed in powers of ρ. Now the equations of motion for the ω_r and Ω_r are

$$\dot{\omega}_r = -\frac{\partial R'''}{\partial \Omega_r}, \quad \dot{\Omega}_r = \frac{\partial R'''}{\partial \omega_r} \quad (r = 1, 2, \ldots, 2n) \tag{36}$$

In order to arrive at the expansions in powers of ρ, first put

$$\Omega_r = \rho\tilde{\Omega}_r \quad (r = 1, 2, \ldots, 2n)$$

and

$$R''' = \rho\tilde{R}, \tag{37}$$

so that the equations of motion for the ω_r and the $\tilde{\Omega}_r$ are

$$\dot{\omega}_r = -\frac{\partial \tilde{R}}{\partial \tilde{\Omega}_r}, \quad \dot{\tilde{\Omega}}_r = \frac{\partial \tilde{R}}{\partial \omega_r} \quad (r = 1, 2, \ldots, 2n),$$

and we may think of $-\tilde{R}$ as the Hamiltonian function for the ω_r and the $\tilde{\Omega}_r$. Consider the expansion of $\tilde{R}$ in powers of ρ,

$$\tilde{R} = \sum_{p=0}^{\infty} \tilde{R}_p,$$

where $\tilde{R}_p$ is of degree p in ρ, and so

$$\tilde{R}_0 = -\sum_{r=1}^{2n} A_r\tilde{\Omega}_r,$$

where we recall that $A_r = O(\varepsilon)$.

Now we may carry out a Lie series type transformation

$$(\omega, \tilde{\Omega}) \mapsto (\omega^*, \tilde{\Omega}^*)$$

using a function $\tilde{T}$ still to be chosen, the transformation being defined by

$$\omega_r = \omega_r^* + \frac{\partial \tilde{T}}{\partial \tilde{\Omega}_r^*} + \sum_{p=1}^{\infty} \frac{1}{(p+1)!} \mathscr{L}_{\tilde{T}}^{p}\left(\frac{\partial \tilde{T}}{\partial \tilde{\Omega}_r^*}\right)$$

and

$$\tilde{\Omega}_r = \tilde{\Omega}_r^* - \frac{\partial \tilde{T}}{\partial \omega_r^*} - \sum_{p=1}^{\infty} \frac{1}{(p+1)!} \mathscr{L}_{\tilde{T}}^{p}\left(\frac{\partial \tilde{T}}{\partial \omega_r^*}\right), \quad (r = 1, 2, \ldots, 2n) \tag{38}$$

so that the equations of motion for the ω_r^* and $\tilde{\Omega}_r^*$ are

$$\dot{\omega}_r^* = -\frac{\partial \tilde{R}^*}{\partial \tilde{\Omega}_r^*}, \dot{\tilde{\Omega}}_r^* = \frac{\partial \tilde{R}^*}{\partial \omega_r^*} \quad (r = 1, 2, \ldots, 2n), \tag{39}$$

where

$$\tilde{R}^*(\omega^*; \tilde{\Omega}^*) \equiv \tilde{R}(\omega; \tilde{\Omega})$$

$$= \tilde{R}(\omega^*, \tilde{\Omega}^*) + \sum_{p=1}^{\infty} \frac{1}{p!} \mathscr{L}_{\tilde{T}}^{p}(\tilde{R}). \tag{40}$$

We now seek expansions

$$\tilde{R}^* = \sum_{p=0}^{\infty} \tilde{R}^*_p, \quad \tilde{T} = \sum_{p=1}^{\infty} \tilde{T}_p \tag{41}$$

in which each of $\tilde{R}^*_p$ and $\tilde{T}_p$ is of degree p in ρ, and will choose the $\tilde{T}_p$ to leave $\tilde{R}^*$ independent of the ω^*_r. Equating terms independent of ρ in Equation (40) gives

$$\tilde{R}^*_0 = \tilde{R}_0(\tilde{\Omega}^*) = -\sum_{r=1}^{2n} A_r \tilde{\Omega}^*_r$$

and equating terms of order ρ gives

$$\begin{aligned} \tilde{R}^*_1 &= \tilde{R}_1 + \{\tilde{R}_0, \tilde{T}_1\} \\ &= \tilde{R}_1 + \sum_{r=1}^{2n} A_r \frac{\partial \tilde{T}_1}{\partial \omega^*_r}. \end{aligned}$$

Corresponding to each term $K_\nu \cos(\sum_r \nu_r \omega_r)$ of $\tilde{R}_1$, we include in $\tilde{T}_1$ the term $K_\nu \sin(\sum_r \nu_r \omega^*_r)/\sum_r \nu_r A_r$, so that $\tilde{R}^*_1$ is left as the sum of those terms of $\tilde{R}_1$ which are independent of the ω^*_r. We note K_ν is of degree one in ρ and at least one in ε, so that this term in $\tilde{T}_1$ is of degree one in ρ, and at least zero in ε. Then we equate in turn terms of each higher order p in ρ in Equation (40), obtaining

$$\tilde{R}^*_p = \Phi_p + \sum_{r=1}^{2n} A_r \frac{\partial \tilde{T}_p}{\partial \omega^*_r},$$

where Φ_p consists of a sum of terms computed from the $\tilde{R}_q$ for $q \leqslant p$ and the $\tilde{T}_q$ for $q < p$, each of which is known or chosen at that stage, so that $\tilde{T}_p$ may be chosen to be of degree p in ρ and of degree at least zero in ε, and to leave $\tilde{R}^*_p$ independent of the ω^*_r.

Now define $\Omega^*_r = \rho \tilde{\Omega}^*_r$ for $r = 1, 2, \ldots, 2n$, and consider the transformation of the whole system

$$(\lambda''', \omega; \Lambda''', \Omega) \mapsto (\lambda^*, \omega^*, \Lambda^*, \Omega^*), \tag{42}$$

where $\Lambda^*_j = \Lambda'''_j$ for $j = 1, 2, \ldots, n$. Put

$$T(\omega^*; \Lambda^*, \Omega^*) \equiv \rho \tilde{T}.$$

Then for any function f of the ω^*, Λ^*, and Ω^*, we have, writing

$$\begin{aligned} f(\omega^*; \Lambda^*, \Omega^*) &\equiv \tilde{f}(\omega^*; \Lambda^*, \tilde{\Omega}^*), \\ \{\tilde{f}, \tilde{T}\} &= \sum_{r=1}^{2n} \left(\frac{\partial \tilde{f}}{\partial \omega^*_r} \frac{\partial \tilde{T}}{\partial \tilde{\Omega}^*_r} - \frac{\partial \tilde{f}}{\partial \tilde{\Omega}^*_r} \frac{\partial \tilde{T}}{\partial \omega^*_r} \right) \\ &= \sum_{r=1}^{2n} \left(\frac{\partial f}{\partial \omega^*_r} \frac{\partial T}{\partial \Omega^*_r} - \rho \frac{\partial f}{\partial \Omega^*_r} \frac{1}{\rho} \frac{\partial T}{\partial \omega^*_r} \right) \\ &= \{f, T\}, \end{aligned}$$

that is,

$$\mathscr{L}_{\tilde{T}}(\tilde{f}) = \mathscr{L}_T(f),$$

so that

$$\begin{aligned}\Omega_r &= \rho\tilde{\Omega}_r \\ &= \rho\left\{\tilde{\Omega}_r^* - \frac{\partial\tilde{T}}{\partial\omega_r^*} - \sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_{\tilde{T}}^p\left(\frac{\partial\tilde{T}}{\partial\omega_r^*}\right)\right\} \\ &= \Omega_r^* - \frac{\partial T}{\partial\omega_r^*} - \sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_T^p\left(\frac{\partial T}{\partial\omega_r^*}\right),\end{aligned}$$

and

$$\begin{aligned}\omega_r &= \omega_r^* + \frac{\partial\tilde{T}}{\partial\tilde{\Omega}_r^*} + \sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_{\tilde{T}}^p\left(\frac{\partial\tilde{T}}{\partial\tilde{\Omega}_r^*}\right) \\ &= \omega_r^* + \frac{\partial T}{\partial\Omega_r^*} + \sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_T^p\left(\frac{\partial T}{\partial\Omega_r^*}\right) \quad (r = 1, 2, \ldots, 2n) \end{aligned} \tag{43}$$

so that if we also put

$$\lambda_j''' = \lambda_j^* + \frac{\partial T}{\partial\Lambda_j^*} + \sum_{p=1}^{\infty}\frac{1}{(p+1)!}\mathscr{L}_T^p\left(\frac{\partial T}{\partial\Lambda_j^*}\right) \quad (j = 1, 2, \ldots, n), \tag{44}$$

then Equation (42) defines a Lie series contact transformation. The Hamiltonian function for the new variables is

$$\begin{aligned}H^*(\lambda^*, \omega^*; \Lambda^*, \Omega^*) &\equiv H'''(\lambda''', \omega; \Lambda''', \Omega) \\ &= -\frac{1}{2}\sum_{j=1}^{n}\mu_j^2\beta_j^3/\Lambda_j'''^2 - K_0 - R'''(\omega; \Lambda''', \Omega) \\ &= -\frac{1}{2}\sum_{j=1}^{n}\mu_j^2\beta_j^3/\Lambda_j'''^2 - K_0 - \rho\tilde{R}(\omega; \Lambda''', \Omega) \\ &= -\frac{1}{2}\sum_{j=1}^{n}\mu_j^2\beta_j^3/\Lambda_j^{*2} - K_0 - \rho\tilde{R}^*(\omega^*; \Lambda^*, \tilde{\Omega}^*) \\ &= -\frac{1}{2}\sum_{j=1}^{n}\mu_j^2\beta_j^3/\Lambda_j^{*2} - K_0 - R^*(\Lambda^*, \Omega^*), \\ &\quad \text{say (using (40))}\end{aligned} \tag{45}$$

where $R^*(\Lambda^*, \Omega^*) \equiv \rho\tilde{R}^*(\omega_r^*; \Lambda^*, \Omega^*)$, which is independent of the ω_r^*. All of the co-ordinates, λ_j^*, and ω_r^*, are now ignorable, so that they are linear functions of the time in the motion represented by this Hamiltonian function, and their conjugate momenta Λ_j^* and Ω_r^*, respectively, are constants of that motion. The formal series representing the solution of the original problem for the Keplerian elements λ_j, $\xi_j, q_j, \Lambda_j, \eta_j$, and p_j, is then given by the explicit expressions (43), (34), (31), (30) (iv), and (17) in turn. Thus each of these elements is represented by a series, in ascending powers of both ε and ρ, composed entirely of terms each of which is the product of a function of the constant momenta Λ_j^* and Ω_r^* and the sine or cosine of a linear combination of the λ_j^* and the ω_r^*, that is, of a linear function of the time. Since $a_j = \Lambda_j^2/(\mu_j\beta_j^2)$, for each j, this also applies to the major semi-axes a_j.

Acknowledgement

I am grateful to Dr L. Duriez for helpful criticisms of an earlier version of this paper.

References

Arnol'd, V. I.: 1963, *Russian Mathematical Surveys* **18**, 85.
Duriez, L.: 1977, *Astron. Astrophys.* **54**, 93.
Duriez, L.: 1978, *Astron. Astrophys.* **68**, 199.
Duriez, L.: 1979, Thesis presented to Universite des Sciences et Techniques de Lille.
Hori, G.: 1966, *Publ. Astron. Soc. Japan* **18**, 287.
Lagrange, J. L.: 1781, *Nouveaux Mem. de l'Acad. Roy. de Sci. et Belles-Ll. de Berlin. see Oevres* **V**, 123.
Message, P. J.: 1976, V. Szebehely and B. Tapley (eds.), *Long-Time Predictions in Dynamics*, D. Reidel Publ. Co., pp. 279–293.
Newcomb, S.: 1874, Smithsonian Contrib. No. 281.
Poincaré, H.: 1893, *Méthodes Nouvelles de la Mécanique Céleste*, Vol. II, Gauthier–Villars, Paris.