University of Glasgow | School of Physics & Astronomy

Astronomy 345: Plasma Theory and Diagnostics II

Session 2021-22

11 Lectures, starting January 2022

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Lecture notes and example problems - https://moodle2.gla.ac.uk

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Contents

1 Plasma descriptions and plasma particle motion . . . 3
2 Uniform, static magnetic and electric fields . . . 18
3 Non-uniform B-field . . . 34
4 Curvature drift . . . 47
5 Magnetic mirroring, adiabatic invariants . . . 60
6 Radiation from charged particles . . . 78
7 Faraday rotation . . . 94
8 Coulomb collisions . . . 108
9 Collisional resistivity and diffusion in plasma . . . 121
10 Kinetic description of plasmas . . . 132
11 Electron plasma waves and Landau damping . . . 143

1 Plasma descriptions and plasma particle motion

PTD II course main topics:

To introduce students to the plasma processes in astrophysical plasma:

- **Orbit theory**: single particle motion, gyration, drifts
- **Radiation by an accelerated charge**: Larmor formula, cyclotron and synchrotron emission
- **Diffusion and resistivity**: collisions in plasma, collision rate, diffusion
- **Plasma kinetics**: kinetic equations, Langmuir waves and Landau dumping

1.1 Recommended literature and useful resources

Introduction to plasma physics and controlled fusion. Volume 1, Plasma physics by Francis F. Chen (2006) [GU Library link] [Amazon Link]
and some elements from these books:
High Energy Astrophysics (Volume 1 and 2) by Malcolm S. Longair [GU library link] [Amazon Link]
The Classical Theory of Fields: Volume 2 (Course of Theoretical Physics) by Landau and Lifshitz [GU library] [Amazon Link]
Physical Kinetics: Volume 10 (Course of Theoretical Physics) by Pitaevskii and Lifshitz [GU library] [Amazon Link]
Some useful on-line books:
Astrophysical Plasmas -Online Book by Steven J Schwartz, David Burgess, and Chris Owen
Plasma Physics - online book by Richard Fitzpatrick

1.2 Description of plasmas

Following *Langmuir, (1923)* we call **plasma** a fully ionized gas. However, nowadays this term became more broader and any media with charged particles can be called a plasma.

Generally a system of N charged particles can be formally described by the system of N equations of motion:

$$\frac{d\vec{\mathbf{p}}_i}{dt} = \vec{\mathbf{F}}(\vec{\mathbf{p}}_1, ..., \vec{\mathbf{p}}_N; \vec{\mathbf{r}}_1, ..., \vec{\mathbf{r}}_N) \tag{1.1}$$

where $\vec{\mathbf{F}}(\vec{\mathbf{p}}_1, ..., \vec{\mathbf{p}}_N; \vec{\mathbf{r}}_1, ..., \vec{\mathbf{r}}_N)$ is the total force acting on i-th particle, $\vec{\mathbf{r}}_i$, $\vec{\mathbf{p}}_i$ are position and momentum of i-th particle. Note that $\vec{\mathbf{F}}$ depends on the positions and velocities of $N-1$ particles.

However, as soon as $N \gg 1$ (e.g. $N \sim 10^{23}$ in one mole of gas !), the solution of the system of 2-nd order differential equations is prohibitively difficult to find and alternative methods of description should be involved.

1.3 There are different types of plasma description

- **Test particles**: solution of equations of motion for $M \ll N$ particles assuming some prescribed force $\vec{\mathbf{F}}(\vec{\mathbf{r}}, t)$, e.g. Lorentz force given by fields $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$, $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$.

- **Fluid description of plasma** (MHD) when plasma is assumed to be a continues media at $L >> l$, where L scale of plasma processes, and l is the mean free path of a particle in a plasma. The plasma fluid is characterised by macroscopic parameters, e.g. density $\rho(\vec{\mathbf{r}}, t)$, fluid velocity $\vec{\mathbf{u}}(\vec{\mathbf{r}}, t)$, pressure $p(\vec{\mathbf{r}}, t)$, etc

- **Kinetic** theory introducing statistical tool to deal with plasma; Particles are described in terms of particle distribution functions $f_{i,e}(\vec{\mathbf{p}}, \vec{\mathbf{r}}, t)$

1.4 Introduction to particle orbit theory

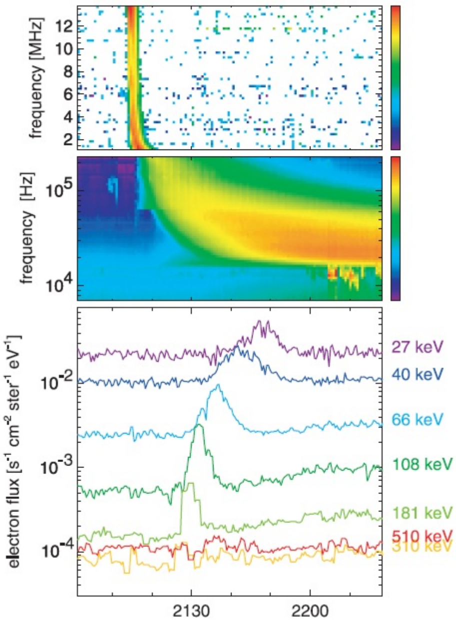


Figure 1.1: *Solar radio emission and energetic electrons near the Earth Krucker et al, 2007*

When the number of particles of interest is small, only single-particle trajectories need to be considered, (i.e. **collective effects** are unimportant). The equation of motion for a **non-relativistic particle**:

$$m\frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

where m is the mass of the particle, q is the charge. $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are assumed to be given and not affected by the particles (e.g. small number of energetic particles from the Sun, see Figure 1.1).

1.5 Constant magnetic field

Particle motion in a constant magnetic field $\vec{\mathbf{B}}$ and $\vec{\mathbf{E}} = 0$. In the absence of an electric field, motion (**non-relativistic**) of a single particle of charge q, mass m, in a magnetic field $\vec{\mathbf{B}}$ is given by:

$$m\frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{1.2}$$

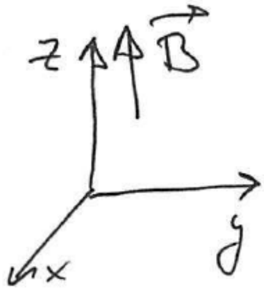
We can demonstrate that **no work** is done on the particle by $\vec{\mathbf{B}}$, and therefore that its kinetic energy remains constant. Indeed, multiplying both parts of Equation (1.2) by $\vec{\mathbf{v}}$:

$$m\vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\underbrace{\left(\frac{m\vec{\mathbf{v}}^2}{2}\right)}_{\text{kinetic energy}} = q\vec{\mathbf{v}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) = 0 \tag{1.3}$$

where the RHS of (1.3) is zero because $\vec{\mathbf{v}} \perp (\vec{\mathbf{v}} \times \vec{\mathbf{B}})$.

Hence the particle's **kinetic energy remains constant.**

1.6 Parallel and perpendicular motion



The particle's motion can be split into components **parallel** and **perpendicular** to $\vec{\mathbf{B}}$. Take $\vec{\mathbf{B}}$ to be in the direction so that $\vec{\mathbf{B}} = B\overrightarrow{\mathbf{e_z}}$ ($\overrightarrow{\mathbf{e_z}}$ is a unit vector.) Recall that

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y & \vec{\mathbf{e}}_z \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = \vec{\mathbf{e}}_x(v_y B - 0) - \vec{\mathbf{e}}_y(v_x B - 0)$$

Since $\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ has no component parallel to $\vec{\mathbf{B}}$ we have

$$\frac{dv_z}{dt} = 0 = \frac{dv_\parallel}{dt} \tag{1.4}$$

Looking at the x and y components of (1.2) we can write

$$\frac{dv_x}{dt} = \frac{q}{m} B v_y \tag{1.5}$$

$$\frac{dv_y}{dt} = -\frac{q}{m} B v_x \tag{1.6}$$

Note: These two equations (1.5,1.6) can also be combined into:

$$\frac{d\vec{\mathbf{v}}_\perp}{dt} = \frac{q}{m} \vec{\mathbf{v}}_\perp \times \vec{\mathbf{B}}\,, \tag{1.7}$$

where $\vec{\mathbf{v}}_\perp$ is the velocity perpendicular to the magnetic field $\vec{\mathbf{B}}$.

1.7 Gyro-motion and gyro-frequency

We solve the Equation (1.5-1.6) by first of all taking the time derivative from both parts of Equations (1.5, 1.6):

$$\frac{d^2 v_x}{dt^2} = \frac{q}{m} B \frac{dv_y}{dt} = -\left(\frac{qB}{m}\right)^2 v_x \tag{1.8}$$

$$\frac{d^2 v_y}{dt^2} = -\frac{q}{m} B \frac{dv_x}{dt} = -\left(\frac{qB}{m}\right)^2 v_y \tag{1.9}$$

Equations (1.8,1.9) describe simple harmonic motion at the cyclotron angular frequency ω_c

Cyclotron or (Gyro-) frequency:
$$\omega_c \equiv \frac{|q|B}{m} \qquad (1.10)$$

$\omega_{ce} = \dfrac{eB}{m_e}$ is the electron cyclotron frequency and $\omega_{cp} = \dfrac{eB}{m_p}$ is the proton cyclotron frequency, $\omega_{ce} \gg \omega_{cp}$ since $m_e \ll m_p$.

1.8 Solution for equation of motion in constant B-field

Solutions of Equation (1.8,1.9) are real parts of

$$v_x = v_{0x} e^{\pm i(\omega_c t + \varphi_x)} \quad \text{and} \quad v_y = v_{0y} e^{\pm i(\omega_c t + \varphi_y)} \tag{1.11}$$

where $\pm$ refers to the sign of the charge $q/|q|$. The constants v_{ox}, v_{oy}, φ_x, φ_y are determined using initial conditions. Consider $v_x(t = 0) = v_0$ then $\varphi_x = 0$ and

$$v_x = v_0 e^{\pm i\omega_c t}$$

Using Equation (1.5) $\dot{v}_x = \frac{q}{m} B v_y$, or $v_y = \frac{m}{qB}\dot{v}_x$, we have

$$v_y = \pm \frac{m v_0}{qB} i\omega_c e^{\pm i\omega_c t}$$

Taking real parts of v_x and v_y, we find

$$v_x = v_0 \cos \omega_c t \quad \text{and} \quad v_y = \mp v_0 \sin \omega_c t \tag{1.12}$$

Forming the product $v_\perp^2 = v_x^2 + v_y^2 = v_0^2(\cos^2\omega_c t + \sin^2\omega_c t) = v_0^2$ gives $v_\perp = v_0$. Velocity components (v_x, v_y) describe **gyromotion** around the z-axis at **angular frequency** ω_c (see Fig 1.3).

The orbit is found by integrating (1.12) with respect to time, so we have

$$x = \frac{v_\perp}{\omega_c}\sin\omega_c t + x_0 \quad \text{and} \quad y = \pm\frac{v_\perp}{\omega_c}\cos\omega_c t + y_0 \tag{1.13}$$

The radius of this orbit, called *Larmor radius*, is

Larmour or (Gyro-) radius $$r_L \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B} \tag{1.14}$$

Meanwhile, there is a uniform component of velocity along $\vec{\mathbf{B}}$, which leads to uniform **guiding centre** motion along $\vec{\mathbf{B}}$ (see Fig 1.3):

$$\frac{dv_z}{dt} = \frac{dv_\parallel}{dt} = 0 \quad \Rightarrow \quad v_z = v_\parallel = \text{const} \quad \Rightarrow \quad z = v_{0z}t + z_0 \tag{1.15}$$

1.9 Typical gyro-frequency and Larmor radius

Electron cyclotron frequency f_{ce} is:

$$f_{ce} = \frac{\omega_{ce}}{2\pi} \simeq 2.8 \left(\frac{B}{1\text{Gauss}} \right) \quad [\text{MHz}]\,,$$

where 1 Gauss$= 10^{-4}$ Tesla.

For plasma particles we can also assume $v_\perp \simeq \sqrt{k_B T / m_{e,p}}$ and $T_e = T_p$:

$$v_{Te} = \sqrt{k_B T / m_e} \simeq 3.8 \times 10^6 \left(\frac{T}{1MK} \right)^{1/2}, \quad \text{m/s}$$

Solar corona (active region): $T \sim 10^6$ K, $B \sim 100$ Gauss, hence we have $f_{ce} \simeq 280$ MHz, $r_{Le} = v_\perp / \omega_{ce} \simeq 2.2 \times 10^{-3}$ m, $r_{Lp} \simeq 43 r_{Le} \simeq 9 \times 10^{-2}$ m.

Solar wind: $T \sim 10^6$ K, $B \sim 5 \times 10^{-9}$ Tesla, hence we have $f_{ce} \simeq 140$ Hz, $r_{Le} = v_\perp / \omega_{ce} \simeq 4.4 \times 10^3$ m, $r_{Lp} \simeq 43 r_{Le} \simeq 1.9 \times 10^5$ m.

Galaxy clusters: $T \sim 10^7$ K, $B \sim 10^{-6}$ Gauss, , hence we have $r_{Le} = \upsilon_\perp/\omega_{ce} \simeq 7 \times 10^5$ m, $r_{Lp} \simeq 43 r_{Le} \simeq 3 \times 10^7$ m.

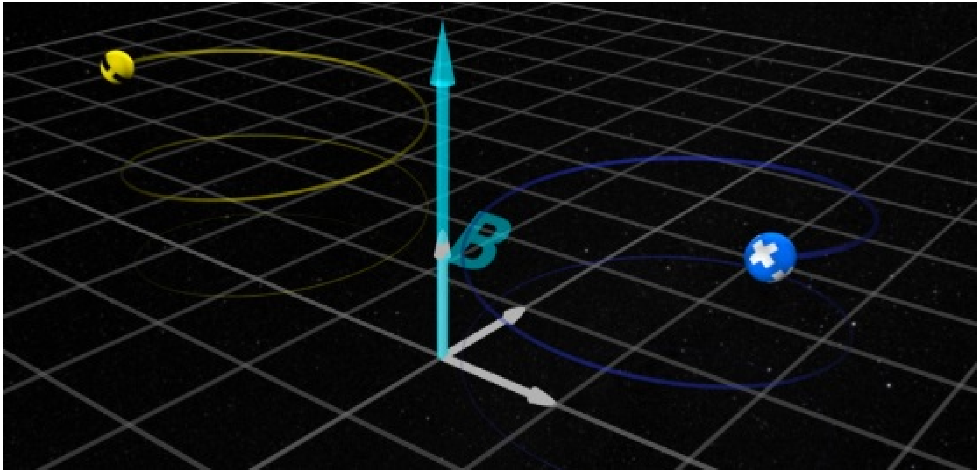


Figure 1.2: *Motion of a charge in constant $\vec{\mathbf{B}}$ field. Animation available from NASA Scientific Visualization Studio*

1.10 Electron and ion motions in uniform $\vec{\mathbf{B}}$

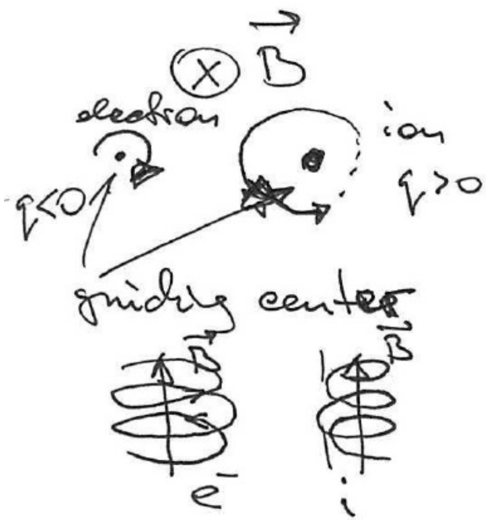


Figure 1.3: *Charge motion in uniform and constant* $\vec{\mathbf{B}}$

The trajectory of a charged particle in space is in general a **helix**: a circular orbit (Eq 1.13) about a guiding centre (x_0, y_0) which is moving along z with v_{0z} (Eq 1.4). However, in other cases, the guiding centre can drift in a direction perpendicular to $\vec{\mathbf{B}}$. Ion Larmor radius $r_{Li} \gg r_{Le}$ since $m_i \gg m_e$. The direction of gyration is such that the magnetic field generated by the charged particles is opposite to the externally imposed field $\vec{\mathbf{B}}$. Hence, it reduces B and plasma is **dia-magnetic**. **Note** that this is the simplest case of charged particle motion, in a uniform constant B-field, with zero electric field. In practise such configurations do not occur in the natural world, and are hard to fabricate in the lab.

2 Uniform, static magnetic and electric fields

LECTURE OUTLINE

- Charge motion in uniform magnetic and electric fields
- Particle acceleration by parallel electric field
- $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ drift
- Parallel and perpendicular charge motion

2.1 Non-zero electric field

When $\vec{\mathbf{E}} \neq 0$ and $\vec{\mathbf{B}} \neq 0$, the equation of motion is :

$$m\frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{2.1}$$

Taking the dot (scalar) product with $\vec{\mathbf{v}}$, as before we have

$$m\vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = q\vec{\mathbf{v}} \cdot \vec{\mathbf{E}} \tag{2.2}$$

The electric field can be presented as a gradient of a potential, $\vec{\mathbf{E}} = -\nabla\phi$, giving, since $\partial\phi/\partial t = 0$

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = -q\vec{\mathbf{v}} \cdot \nabla\phi = -q\frac{d\vec{\mathbf{r}}}{dt} \cdot \frac{\partial\phi}{\partial\vec{\mathbf{r}}} = -q\frac{d\phi}{dt}$$

Rewriting, we have:

$$\frac{d}{dt}\left(\frac{mv^2}{2} + q\phi\right) = 0$$

Hence, the sum of the particle's kinetic and potential energies remains constant in the presence of uniform, constant $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

An electric field **does work** on the charged particle:

- $\vec{\mathbf{E}}_{||}$, the component of electric field parallel to $\vec{\mathbf{B}}$ results in **acceleration of the particle along $\vec{\mathbf{B}}$**
- $\vec{\mathbf{E}}_{\perp}$, the component of electric field perpendicular to $\vec{\mathbf{B}}$ results in the **drift** of a particle (drift of the gyro-centre) acceleration across magnetic field lines.

Note, to maintain an electric field parallel to the magnetic field requires external work to be done, since the conductivity is high and the electrons can move easily along the field to maintain neutrality. For example, $\vec{\mathbf{E}}_{||}$ has been measured in the Earth's ionosphere (e.g. Ergun et al, 2005), where it is thought to be responsible for auroral particle acceleration (see Fig 2.1);

pulsar magnetosphere (see Fig 2.2); electron acceleration by parallel electric fields during magnetic reconnection (e.g. Egedal et al, 2012)

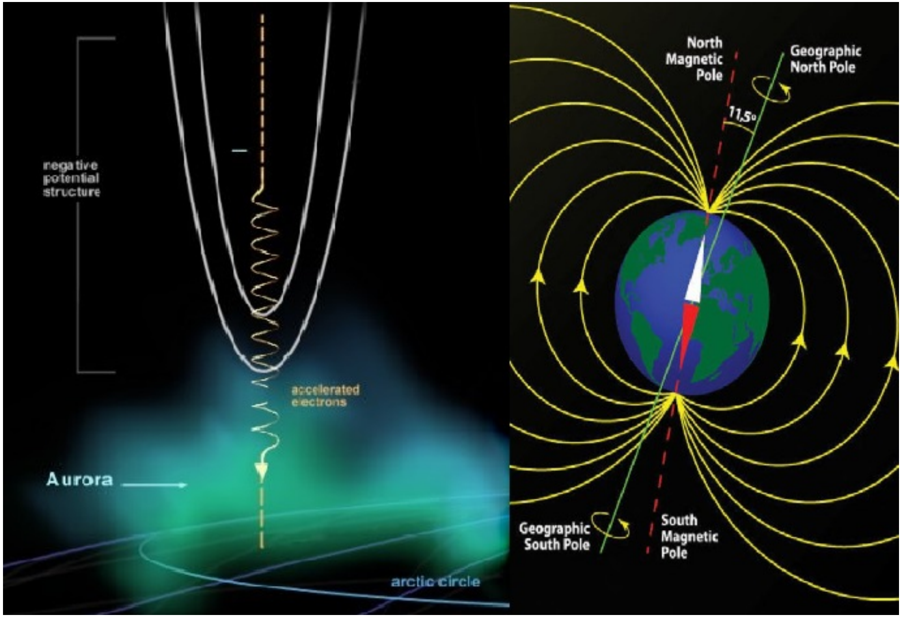


Figure 2.1: *Artistic view of electrons, responsible for aurora, spiralling down magnetic field lines. The U-shaped potential structure illustrates the region where electrons get accelerated on their way down to the upper atmosphere where they collide with neutral atoms and molecules, which in turn produce the aurora light show. from ESA webpage*

2.2 Example: Electric field in pulsar magnetosphere

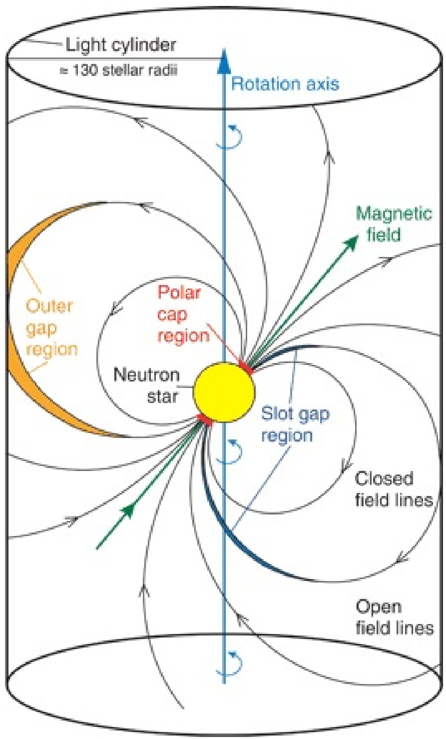


Figure 2.2: *A sketch of pulsar magnetosphere. Particles are accelerated along the magnetic field lines (e.g. in the polar cap region) and emit electromagnetic radiation via the synchrotron-curvature mechanism. from MAGIC Collaboration*

2.3 Electric field acceleration

Consider $\vec{\mathbf{B}} = B\overrightarrow{\mathbf{e_z}}$ and $\vec{\mathbf{E}} = (E_x, 0, E_z)$, i.e. $\vec{\mathbf{E}}$ has a component **parallel** and **perpendicular** to $\vec{\mathbf{B}}$.

Looking at components of (2.1) we have in cartesian coordinates:

$$\begin{aligned} m\frac{dv_z}{dt} &= qE_z + 0 \\ m\frac{dv_x}{dt} &= qE_x + qBv_y \\ m\frac{dv_y}{dt} &= 0 - qBv_x \end{aligned} \tag{2.3}$$

The first equation corresponds to acceleration along the magnetic field direction and one finds:

$$v_z(t) = v_{0z} + \frac{qE_{\|}t}{m}$$

Differentiating the equations (2.3) for υ_x, υ_y, one finds for constant $\vec{\mathbf{E}}$

$$\ddot{\upsilon}_x = -\omega_c^2 \upsilon_x \tag{2.4}$$

$$\ddot{\upsilon}_y = -\frac{qB}{m}\dot{\upsilon}_x = -\frac{qB}{m}\left(\frac{q}{m}E_x + \frac{qB}{m}\upsilon_y\right) \tag{2.5}$$

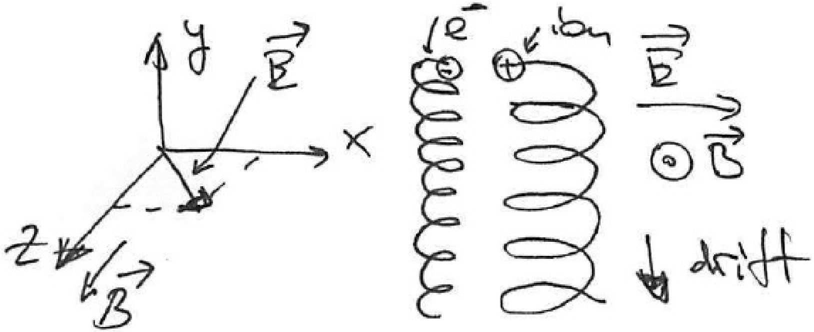
Rewriting (2.5), we have

$$\ddot{\upsilon}_y = -\omega_c^2\left(\frac{E_x}{B} + \upsilon_y\right) \quad \Rightarrow \quad \frac{d^2}{dt^2}\left(\frac{E_x}{B} + \upsilon_y\right) = -\omega_c^2\left(\frac{E_x}{B} + \upsilon_y\right)$$

oscillations for $E_x/B + \upsilon_y$.

Similar to (1.11), the solution of Equation (2.4,2.5) with initial condition $\upsilon_x(t = 0) = \upsilon_\perp$ is

$$\begin{aligned}\upsilon_x &= \upsilon_\perp e^{\pm i\omega_c t}\\ \upsilon_y &= \pm i\upsilon_\perp e^{\pm i\omega_c t} - \frac{E_x}{B}\end{aligned}$$



Hence, the Larmor motion is the same as for $\vec{\mathbf{E}} = 0$, but there is a drift of the guiding centre in the $-\overrightarrow{\mathbf{e_y}}$ direction for $E_x > 0$.

The **drift** appears in perpendicular direction, because when proton moves in the direction of $\vec{\mathbf{E}}$ then $\vec{\mathbf{v}}_\perp$ increases, so r_L increases. When proton moves anti-parallel to $\vec{\mathbf{E}}$ then $\vec{\mathbf{v}}_\perp$ and r_L both decrease. Hence, gyro-orbits are therefore not closed circles; instead the guiding centre drifts in y-direction.

2.4 ExB drift velocity

To obtain a general formula for the velocity of the guiding centre, we start with equation of motion

$$m\frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

and take a cross product with $\vec{\mathbf{B}}$

$$m\vec{\mathbf{B}} \times \frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{B}} \times \vec{\mathbf{E}} + q\vec{\mathbf{B}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})$$

Since we are interested in the drift with constant speed, i.e. $\dot{\vec{\mathbf{v}}} = 0$, then:

$$0 = \vec{\mathbf{B}} \times \vec{\mathbf{E}} + \vec{\mathbf{B}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \times \vec{\mathbf{E}} + \vec{\mathbf{v}}B^2 - \vec{\mathbf{B}}(\vec{\mathbf{v}} \cdot \vec{\mathbf{B}})$$

Taking only transverse component of $\vec{\mathbf{v}}$, i.e. $\vec{\mathbf{v}} \cdot \vec{\mathbf{B}} = 0$, we find

$$\vec{\mathbf{v}}_E = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{B^2} \qquad (2.6)$$

which is **the electric field drift or $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$-drift** of the guiding centre.

The drift velocity is directed perpendicular to both $\vec{\mathbf{B}}$ and $\vec{\mathbf{E}}$.

It is important to note that $\vec{\mathbf{v}}_E$ is independent of q, m, and $\vec{\mathbf{v}}$.

2.5 ExB drift of electrons and ions

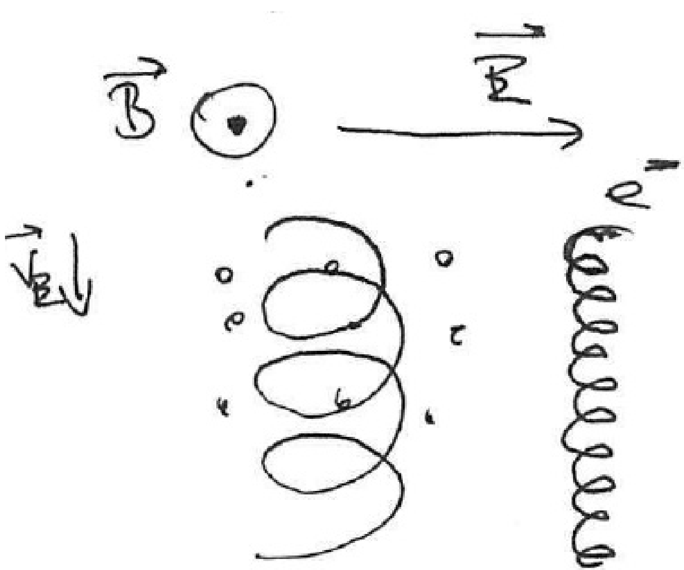


Figure 2.3

Let us consider why $\vec{\mathbf{v}}_E$ is independent of q, m, and v.

As the particle moves in circular orbit, it **gains** energy from $\vec{\mathbf{E}}$ and increases $\vec{\mathbf{v}}_\perp$, hence r_L (Figure 2.3). In the second half-cycle, it **loses** energy and decreases r_L. This difference in r_L on the left and right sides of the orbit causes the $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ drift.

Electron gyrates in the opposite to ion's direction with smaller r_L and hence less per cycle. However, the frequency ω_L is also larger by the same amount, so per unit of time these **two effects exactly cancel**.

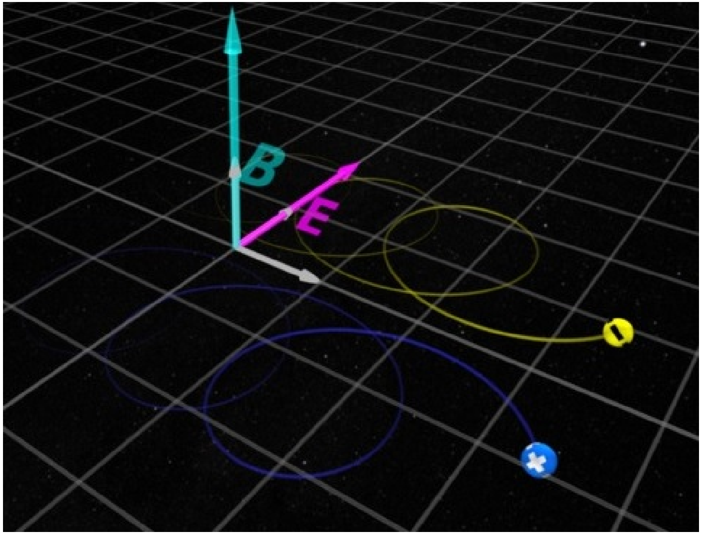


Figure 2.4: *Electron and proton $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ drifts. Visualisation is from NASA scientific visualisation studio.*

2.6 Parallel and perpendicular motion

Let us consider $\|$ and $\perp$ motion of a charge. We can split the motion into two parts

$$\vec{\mathbf{v}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}_E \,,$$

where $\vec{\mathbf{v}}_E = \vec{\mathbf{E}} \times \vec{\mathbf{B}}/B^2$ is **the constant drift velocity of the gyro-centre.** Then

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{d\vec{\mathbf{u}}}{dt} + \frac{d\vec{\mathbf{v}}_E}{dt} = \frac{d\vec{\mathbf{u}}}{dt} + 0$$

From equation of motion (1.2), we have

$$m\frac{d\vec{\mathbf{u}}}{dt} = q\vec{\mathbf{E}} + q\vec{\mathbf{u}} \times \vec{\mathbf{B}} + q\frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{B^2} \times \vec{\mathbf{B}}$$

and using $(\vec{\mathbf{E}} \times \vec{\mathbf{B}}) \times \vec{\mathbf{B}} = \vec{\mathbf{B}}(\vec{\mathbf{E}} \cdot \vec{\mathbf{B}}) - \vec{\mathbf{E}}B^2$, the equation of motion becomes

$$m\frac{d\vec{\mathbf{u}}}{dt} = q\vec{\mathbf{E}} + q\vec{\mathbf{u}} \times \vec{\mathbf{B}} + q\frac{\vec{\mathbf{E}} \cdot \vec{\mathbf{B}}}{B^2}\vec{\mathbf{B}} - q\vec{\mathbf{E}} = q\vec{\mathbf{u}} \times \vec{\mathbf{B}} + q(\vec{\mathbf{E}} \cdot \vec{\mathbf{b}})\vec{\mathbf{b}} \,,$$

where $\vec{\mathbf{b}} = \vec{\mathbf{B}}/B$.

Hence the components of $\vec{\mathbf{u}}$ are

$$\begin{aligned} m\frac{du_{\parallel}}{dt} &= qE_{\parallel} \\ m\frac{d\vec{\mathbf{u}}_{\perp}}{dt} &= q\vec{\mathbf{u}}_{\perp} \times \vec{\mathbf{B}} \end{aligned}$$

where the first equation describes evolution of $u_{\parallel}$ and implies $\parallel$ motion with uniform acceleration driven by $E_{\parallel}$.

The second equation describes $\vec{\mathbf{u}}_{\perp}$ and leads to uniform circular motion $\perp$ to magnetic field (gyro-motion).

2.7 External force drift

The result of $\vec{\mathbf{E}}$ drift can be applied to other forces by replacing $q\vec{\mathbf{E}}$ in the equation of motion by a general force $\vec{\mathbf{F}}$:

$$m\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{F}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

Again, we have that the motion is the sum of three parts: acceleration parallel to $\vec{\mathbf{B}}$, gyration perpendicular to $\vec{\mathbf{B}}$,

and a uniform drift at velocity $\vec{\mathbf{v}}_F$, with

$$\vec{\mathbf{v}}_F = \frac{1}{q}\frac{\vec{\mathbf{F}} \times \vec{\mathbf{B}}}{B^2} \tag{2.7}$$

$\vec{\mathbf{v}}_F$-drift is perpendicular to $\vec{\mathbf{F}}$ and $\vec{\mathbf{B}}$, but unlike $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$-drift, depends on the particle charge.

2.8 Gravitational force drift

When the external force is gravity $\vec{\mathbf{F}} = m\vec{\mathbf{g}}$, we have

$$\vec{\mathbf{v}}_g = \frac{m}{q}\frac{\vec{\mathbf{g}} \times \vec{\mathbf{B}}}{B^2} \tag{2.8}$$

So under the influence of a gravitational force, **ions and electrons drift in opposite directions**,[1] so there is a net current in such plasma.

The physical reason for this drift is again the change in Larmor radius as the particle gains and loses energy in the field.

[1] Compare to $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ drift and note the similarities and differences

3 Non-uniform B-field

LECTURE OUTLINE

- Charge motion in non-uniform magnetic field
- Grad-B drift velocity

3.1 Nonuniform magnetic fields

Magnetic fields are inhomogeneous (e.g. galactic Fig 3.1, solar Fig 3.2)

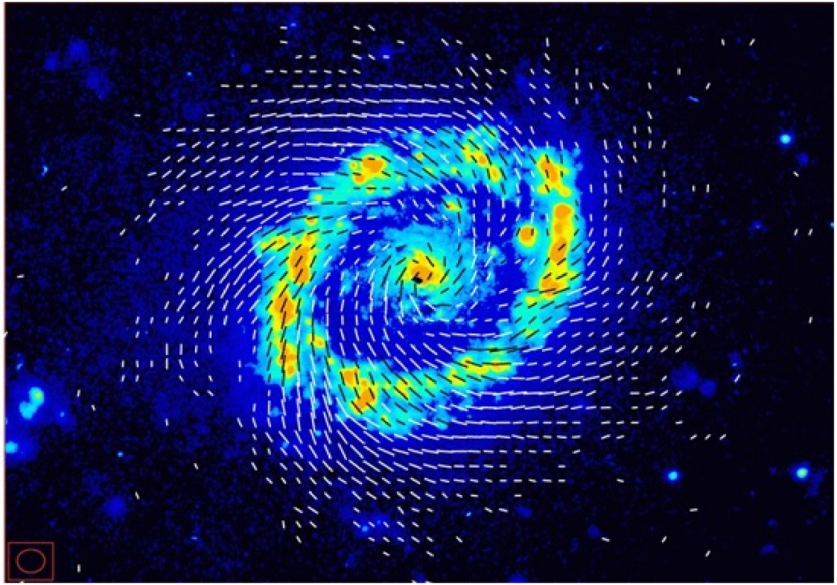


Figure 3.1: *The image above shows an example of a spiral magnetic field of the galaxy NGC 4736. Chyzy & Buta 2008.*

3.2 Motion in non-uniform B-field

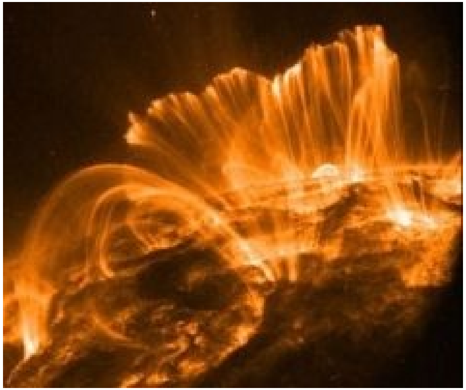


Figure 3.2: *Solar loops, image SOHO/EIT*

As soon as we introduce inhomogeneity, i.e. $\vec{\mathbf{B}}(\vec{\mathbf{r}})$, the problem of particle motion becomes too complicated to solve exactly. To get an approximate answer we can use **the small parameter, $r_L/L \ll 1$**, where r_L is the Larmor radius and L is the scale length of the inhomogeneity.

For example, particles in the solar corona $T \simeq 2$ MK, $B = 10^{-2}$ Tesla ($=$ 100 Gauss) $\upsilon_{Te} = \sqrt{k_b T/m_e} \simeq 5 \times 10^6$ m/s, $r_{Le} = \upsilon_{Te}/\omega_{pe} \simeq 3 \times 10^{-3}$ m; $\upsilon_{Ti} = \sqrt{k_b T/m_i} \simeq 1.3 \times 10^5$ m/s, $r_{Li} = \upsilon_{Ti}/\omega_{pe} \simeq 1.3 \times 10^{-1}$ m; Hence at the scales $\gg 1.3 \times 10^{-1}$ m, small Larmor radius approximation is applicable. Such approximation is valid in many astrophysical settings, but not always.

Let us consider inhomogeneity of $\vec{\mathbf{b}}$ and B separately.

3.3 Grad-B drift

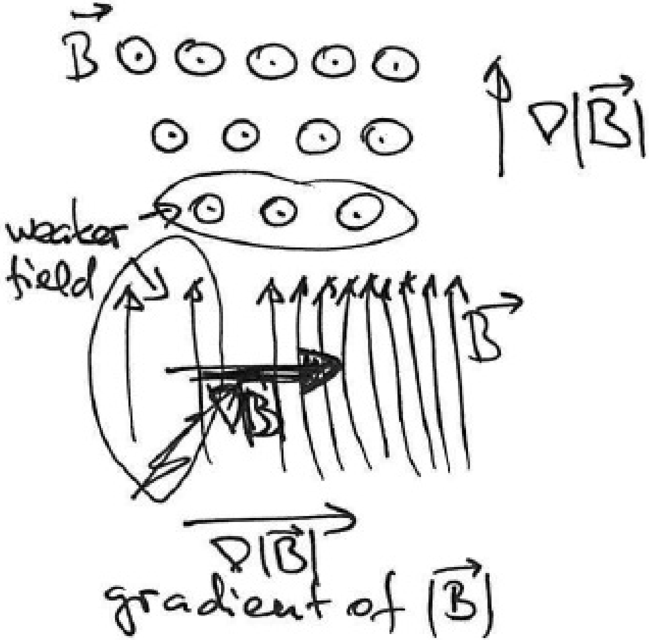


Figure 3.3: *Non-uniform* B

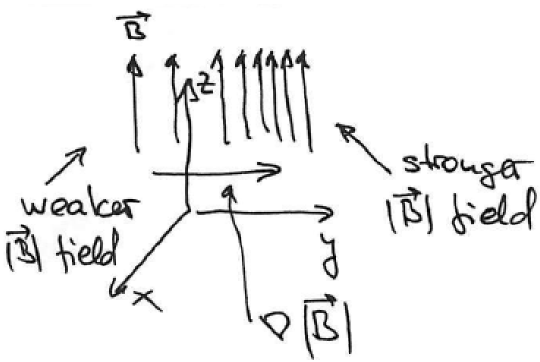
We consider the plasma with the change of B (Figure 3.3), while the direction of $\vec{\mathbf{B}}$ does not change and we also consider $\nabla B \perp \vec{\mathbf{B}}$.

Consider the Lorentz force

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

when $\vec{\mathbf{B}}$ is constant, the force F averaged over a gyration is zero, since the particle spends same amount of time in each direction. In case of weakly inhomogeneous field $\vec{\mathbf{B}}$ in Taylor series, using that the particle position

$\vec{\mathbf{r}}_0 + \vec{\mathbf{r}}$, where $\vec{\mathbf{r}}_0$ is the position of gyro-center, and $\vec{\mathbf{r}}$ is the particle motion around the gyro-centre.



Let us consider $\vec{\mathbf{B}} = (0,0,B_z)$ and $B_z = B_z(y)$. Then the force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \vec{\mathbf{F}}$ could be written:

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y & \vec{\mathbf{e}}_z \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix} = \vec{\mathbf{e}}_x v_y B_z - \vec{\mathbf{e}}_y v_x B_z + 0$$

Let us take the expansion of $\vec{\mathbf{B}}(\vec{\mathbf{r}}_0 + \vec{\mathbf{r}})$, $r \ll r_0$ near gyrocentre $\vec{\mathbf{r}}_0$ (You can consider x, y, z components and show that for vector field $\vec{\mathbf{A}}(\vec{\mathbf{r}}_0 + \vec{\mathbf{r}})$, we can write

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}_0 + \vec{\mathbf{r}}) = \vec{\mathbf{A}}(\vec{\mathbf{r}}_0) + (\vec{\mathbf{r}} \cdot \nabla)\vec{\mathbf{A}}(\vec{\mathbf{r}}_0) + \dots,$$

when $\vec{\mathbf{r}} \ll \vec{\mathbf{r}}_0$):

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}_0 + \vec{\mathbf{r}}) = \vec{\mathbf{B}}(\vec{\mathbf{r}}_0) + (\vec{\mathbf{r}} \cdot \nabla)\vec{\mathbf{B}}(\vec{\mathbf{r}}_0) + \ldots$$
$$B_z(y_0 + y) = B(y_0) + y\frac{\partial}{\partial y}B_z + \ldots$$

then we have:

$$F_x = qv_y(B_0 + y\frac{\partial B_z}{\partial y})$$
$$F_y = -qv_x(B_0 + y\frac{\partial B_z}{\partial y})$$

Now for x, y, v_x, v_y, let us take the solution for uniform B (Equations 1.12, 1.13)

$$v_x = v_\perp \cos \omega_c t, \qquad v_y = \mp v_\perp \sin \omega_c t$$
$$x = \frac{v_\perp}{\omega_c} \sin \omega_c t, \qquad y = \pm \frac{v_\perp}{\omega_c} \cos \omega_c t$$

Then

$$F_x = \mp q v_\perp \sin\omega_c t \left(B_0 \pm \frac{v_\perp}{\omega_c} \cos\omega_c t \frac{\partial B_z}{\partial y} \right)$$
$$F_y = -q v_\perp \cos\omega_c t \left(B_0 \pm \frac{v_\perp}{\omega_c} \cos\omega_c t \frac{\partial B_z}{\partial y} \right)$$

we seek the force averaged over gyro-period $T = 2\pi/\omega_c$, i.e. $\langle F_x \rangle$, $\langle F_y \rangle$:

$$\langle F_x \rangle \equiv \frac{1}{T} \int_0^T F_x dt$$

since

$$\langle \sin\omega_c t \rangle = 0$$
$$\langle \cos\omega_c t \rangle = 0$$
$$\langle \cos\omega_c t \sin\omega_c t \rangle = 0$$
$$\langle \cos^2\omega_c t \rangle = \frac{1}{2}$$

we can write

$$\langle F_x \rangle = 0$$
$$\langle F_y \rangle = \mp q v_\perp \frac{v_\perp}{\omega_c} \frac{1}{2} \frac{\partial B_z}{\partial y}$$

Since the choice of the y-axis was arbitrary we can write

$$\langle \vec{\mathbf{F}} \rangle = \mp q \frac{v_\perp^2}{\omega_c} \frac{1}{2} \nabla B$$

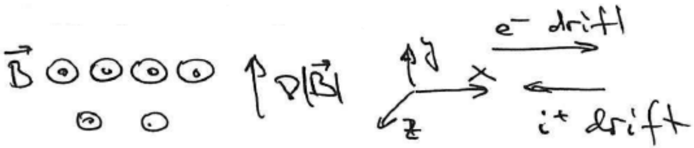
Then the guiding centre drift from Equation (2.7) is

$$\vec{\mathbf{v}}_{\nabla B} = \frac{1}{q} \frac{\langle \vec{\mathbf{F}} \rangle \times \vec{\mathbf{B}}}{B^2} = \mp \frac{1}{q} q \frac{v_\perp^2}{2\omega_c} \frac{\nabla B \times \vec{\mathbf{B}}}{B^2} = \pm \frac{v_\perp^2}{2\omega_c} \frac{\vec{\mathbf{B}} \times \nabla B}{B^2}$$

Hence

grad-B drift velocity: $$\vec{\mathbf{v}}_{\nabla B} = \pm\frac{v_\perp^2}{2\omega_c}\frac{\vec{\mathbf{B}}\times\nabla B}{B^2} \tag{3.1}$$

where factor $1/2$ from averaging, and $\pm$ stands for the charge sign of ions and electrons.



The drift velocity is directed perpendicular to both $\vec{\mathbf{B}}$ and ∇B.

3.4 Animation of charge motion and drift

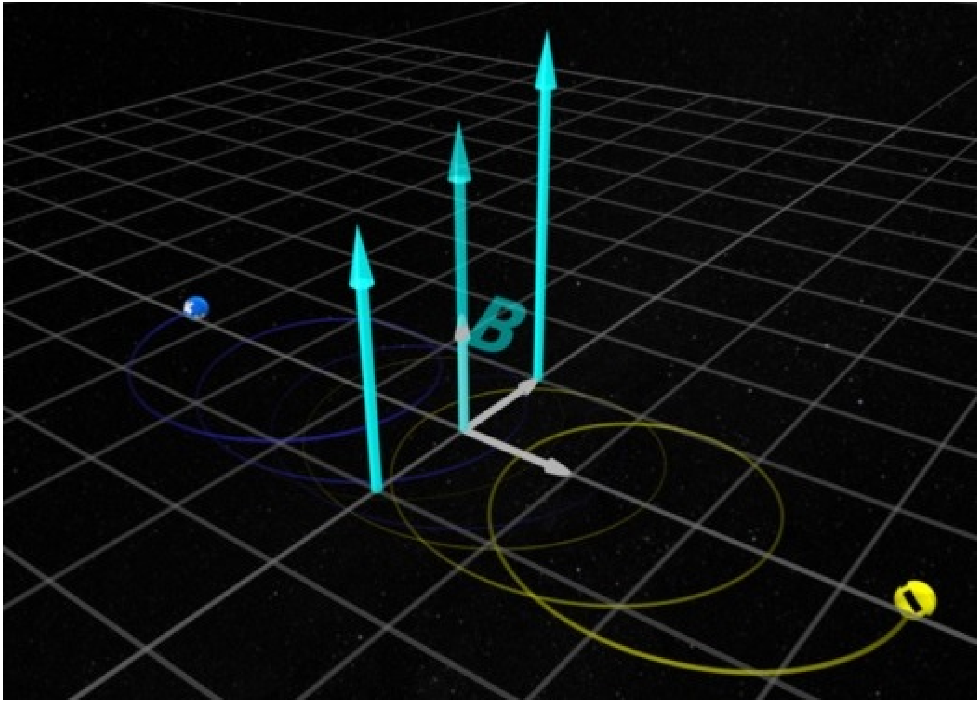


Figure 3.4: *Particle Drift in a Magnetic Gradient. Animation from NASA Scientific Visualization Studio*

3.5 Orbits and Grad-B drift

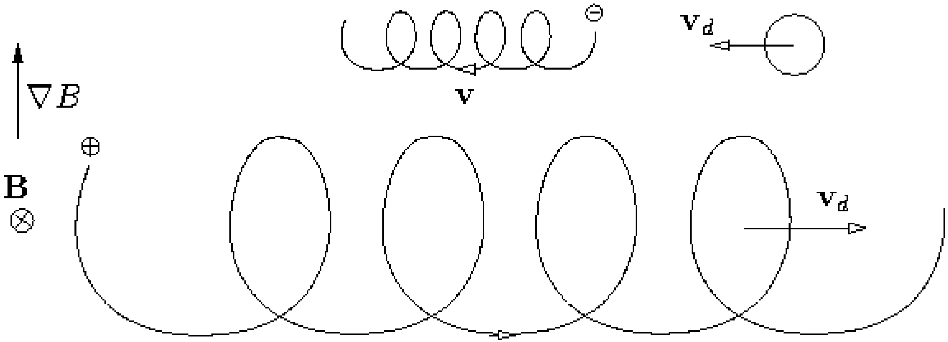


Figure 3.5: *Curvature (inverse of curvature radius) of orbit is greater where B is greater causing orbit to be small on that side. Result is a drift perpendicular to both B and ∇B. Notice, though, that electrons and ions go in* ***opposite*** *directions, unlike $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ drift. Figure from F. Chen book*

3.6 Drift speed

Let us estimate the drift speed. We can write from Equation 3.1:

$$|\vec{\mathbf{v}}_{\nabla B}| \simeq \frac{v_\perp^2}{2\omega_c}\frac{B|\nabla B|}{B^2}$$

Assuming $|\nabla B| \simeq B/L$, where L is the inhomogeneity length, and isotropic velocity distribution $v_x = v_y = v_z = v$:

$$|\vec{\mathbf{v}}_{\nabla B}| \simeq \frac{v}{\omega_c} v \frac{1}{L} \simeq \frac{r_L}{L} v$$

so we see that the drift is r_L/L times the actual particle speed v, recall we assumed that it is a small parameter.

Indeed, the **drift due to the magnetic field inhomogeneity is in general much smaller than the particle speed**.

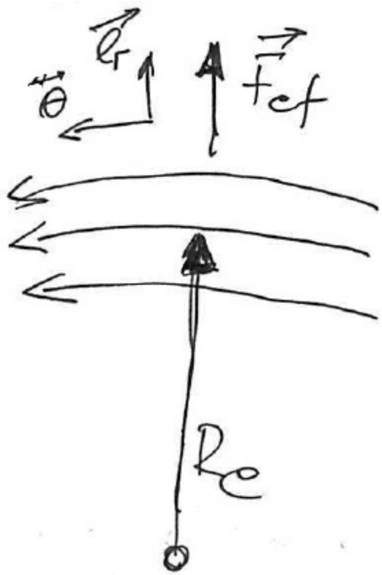
Note that drift-speed derivation is valid when the drift speed is much slower than the particle speed.

4 Curvature drift

LECTURE OUTLINE

- Charge motion in curved magnetic field
- Magnetic field curvature drift velocity
- Curvature and grad-B drifts combined
- Ring current in Earth magnetosphere
- Drifts and particle acceleration

4.1 Curved magnetic fields



Let us assume that $\vec{\mathbf{B}}$ is curved with constant radius R_c and we take B to be constant. Again R_c needs to be large in comparison with r_L.[2]

The guiding centre drift arises from the centrifugal force felt by the particles

$$\vec{\mathbf{F}}_{cf} = \frac{mv_\parallel^2}{R_c}\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = mv_\parallel^2 \frac{\vec{\mathbf{R}}_c}{R_c^2}$$

[2]Strictly speaking such field does not obey Maxwell equation in vacuum, so ∇B should be added too.

4.2 Field direction

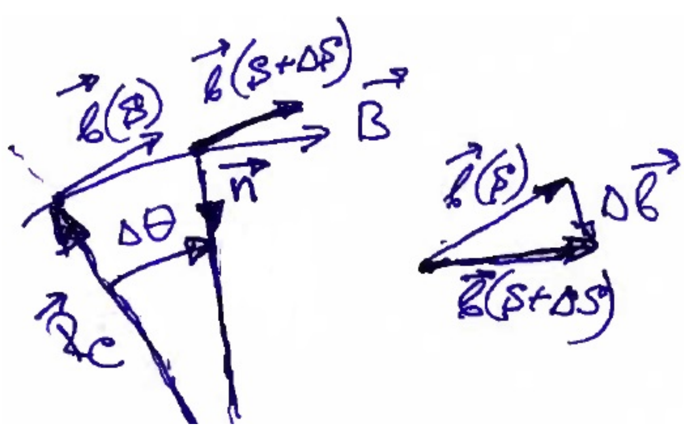
Let us introduce $(\vec{\mathbf{b}} \cdot \nabla) \equiv \frac{\partial}{\partial s}$ is the derivative along the field line, which is the rate of change as one moves in the direction $\vec{\mathbf{b}} = \vec{\mathbf{B}}/B$, then

$$\frac{\partial}{\partial s}\left(B\vec{\mathbf{b}}\right) = B \underbrace{\frac{\partial \vec{\mathbf{b}}}{\partial s}}_{\text{change of } \vec{\mathbf{b}}} + \vec{\mathbf{b}} \underbrace{\frac{\partial B}{\partial s}}_{\text{change of B}}$$

Note that the vector $d\vec{\mathbf{b}}/ds$ is directed perpendicular to $\vec{\mathbf{b}}$ and towards the centre of osculating circle, hence we have minus sign.

Let us calculate the value of $d\vec{\mathbf{b}}/ds$.

4.3 Derivative of a unit vector $d\vec{\mathbf{b}}/ds$



Let us consider the change of unit vector $\vec{\mathbf{b}}$ along the osculating circle of radius R_c. In the limit $\Delta s \to 0$, we can write $\Delta s = R_c \Delta\theta$. When $\vec{\mathbf{b}}$ moves $\Delta\theta$, from the isosceles triangle with $|\vec{\mathbf{b}}(s+\Delta s)| = |\vec{\mathbf{b}}(s)| = 1$, we have $|\vec{\mathbf{b}}(s+\Delta s) - \vec{\mathbf{b}}(s)| = \Delta\theta$, hence

$$\frac{d\vec{\mathbf{b}}}{ds} = \lim_{\Delta s \to 0} \frac{\vec{\mathbf{b}}(s+\Delta s) - \vec{\mathbf{b}}(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{R_c \Delta\theta}\vec{\mathbf{n}} = \frac{\vec{\mathbf{n}}}{R_c} = -\frac{\vec{\mathbf{R}}_c}{R_c^2}.$$

The direction of $d\vec{\mathbf{b}}/ds$ in the limit $\Delta s \to 0$ can be seen from the triangle in the Figure. $\Delta\vec{\mathbf{b}}$ is directed perpendicular to $\vec{\mathbf{b}}$ and along $\vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is the unit normal vector. $d\vec{\mathbf{b}}/ds$ is directed towards the centre of osculating circle, hence we have minus sign.

4.4 Force due to the cuvature

Using the expression for $d\vec{\mathbf{b}}/ds$ and the definition $(\vec{\mathbf{b}} \cdot \nabla) \equiv \frac{\partial}{\partial s}$, we can re-write in $\vec{\mathbf{r}}$ coordinates

$$\frac{\vec{\mathbf{R}}_c}{R_c^2} = -\frac{\partial \vec{\mathbf{b}}}{\partial s} = -\frac{(\vec{\mathbf{b}} \cdot \nabla)\vec{\mathbf{B}}}{B} = -\frac{1}{B^2}(\vec{\mathbf{B}} \cdot \nabla)\vec{\mathbf{B}}$$

Then the force becomes

$$\vec{\mathbf{F}}_{cf} = -\frac{mv_{\parallel}^2}{B^2}(\vec{\mathbf{B}} \cdot \nabla)\vec{\mathbf{B}}$$

and using (Equation 2.7), we substitute the force:

$$\vec{\mathbf{v}}_R = \frac{1}{q}\frac{\vec{\mathbf{F}}_{cf} \times \vec{\mathbf{B}}}{B^2} = \frac{mv_{\parallel}^2}{qB^2}\frac{\vec{\mathbf{B}} \times (\vec{\mathbf{B}} \cdot \nabla)\vec{\mathbf{B}}}{B^2}$$

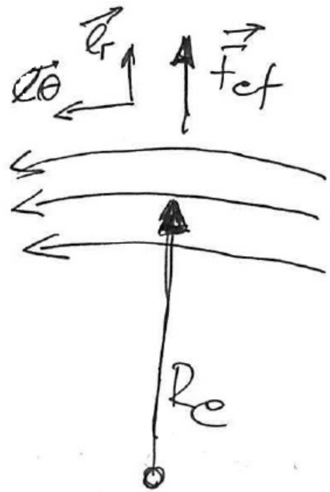
and find the velocity of this drift:

Curvature drift: $$\vec{\mathbf{v}}_R = \pm \frac{v_\parallel^2}{\omega_c} \frac{\vec{\mathbf{R}}_c \times \vec{\mathbf{B}}}{B R_c^2} = \frac{m v_\parallel^2}{q B^4} \left(\vec{\mathbf{B}} \times (\vec{\mathbf{B}} \cdot \nabla) \vec{\mathbf{B}} \right) \qquad (4.1)$$

The curvature drift velocity is directed perpendicular to $\vec{\mathbf{B}}$ and $\vec{\mathbf{R}}_c$ and depends on the charge sign and mass.

4.5 Curvature and grad-B drifts combined

Let us compute grad-B drift which accompanies the curvature drift: when the decrease of B with radius is taken into account, i.e. in cylindrical coordinates $\vec{\mathbf{B}} = (0, B_\theta(r), 0)$.



In vacuum, Maxwell equations require $\nabla \cdot \vec{\mathbf{B}} = 0$ and $\nabla \times \vec{\mathbf{B}} = 0 = \vec{\mathbf{j}}$, where $\vec{\mathbf{j}}$ is the current density. In the cylindrical coordinates $\nabla \times \vec{\mathbf{B}}$ has only $\vec{\mathbf{e}}_z$ component, i.e. out of figure plane. Recall that in cylindrical coordinates

$$\nabla \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{e}}_r & \vec{\mathbf{e}}_\theta & \vec{\mathbf{e}}_z \\ \frac{1}{r}\frac{\partial}{\partial r} r & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ B_r & B_\theta & B_z \end{vmatrix}$$

Since $\vec{\mathbf{B}} = 0\vec{\mathbf{e}}_r + B_\theta \vec{\mathbf{e}}_\theta + 0\vec{\mathbf{e}}_z$, we have

$$(\nabla \times \vec{\mathbf{B}})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

then we find that

$$B_\theta(r) = \frac{\text{const}}{r}$$

hence $B \propto 1/R_c$ and

$$\left.\frac{\nabla B}{B}\right|_{r=R_c} = -\frac{\vec{\mathbf{R}}_c}{R_c^2}$$

Using the grad-B drift velocity (Equation 3.1):

$$\vec{\mathbf{v}}_{\nabla B} = \pm \frac{v_\perp^2}{2\omega_c}\frac{\vec{\mathbf{B}} \times \nabla B}{B^2}$$

we substitute ∇B due to the curvature and find

$$\vec{\mathbf{v}}_{\nabla B} = \mp \frac{v_\perp^2}{2\omega_c}\frac{\vec{\mathbf{B}} \times \vec{\mathbf{R}}_c}{BR_c^2} = \frac{1}{2}\frac{mv_\perp^2}{q}\frac{\vec{\mathbf{R}}_c \times \vec{\mathbf{B}}}{B^2 R_c^2} \tag{4.2}$$

Combining the curvature drift velocity (4.1) and (4.2) we find the velocity of the combined drift

$$\vec{\mathbf{v}}_B = \vec{\mathbf{v}}_{\nabla B} + \vec{\mathbf{v}}_R = \frac{m}{q}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\frac{\overrightarrow{\mathbf{R_c}} \times \vec{\mathbf{B}}}{B^2 R_c^2} \tag{4.3}$$

4.6 Drifts in non-uniform B

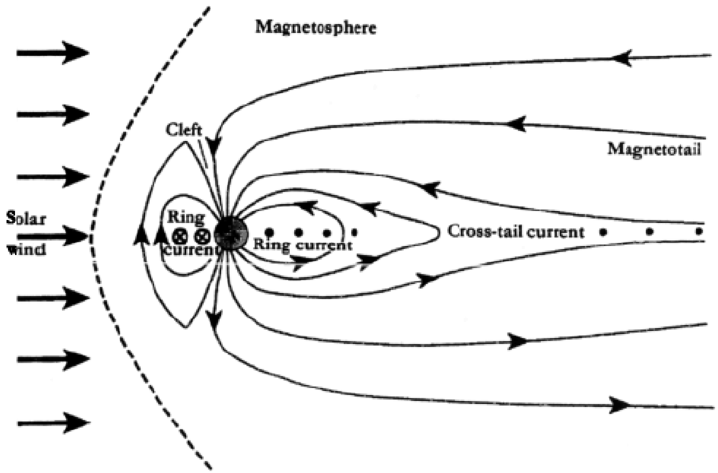


Figure 4.1: *Ring current and Earth magnetosphere, Daglis et al, 1999*

We see that the total drift due to non-uniform field $\vec{\mathbf{B}}$

$$\vec{\mathbf{v}}_B = \frac{m}{q}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\frac{\overrightarrow{\mathbf{R_c}} \times \vec{\mathbf{B}}}{B^2 R_c^2}$$

is dependent on particle charge, mass and velocity. The direction of drift is $\perp$ to both $\vec{\mathbf{R}}_c$ and $\vec{\mathbf{B}}$.

In the dipole field of Earth magnetosphere, we have drift depending on the charge, hence there is a current, called **ring current**.

4.7 Ring current

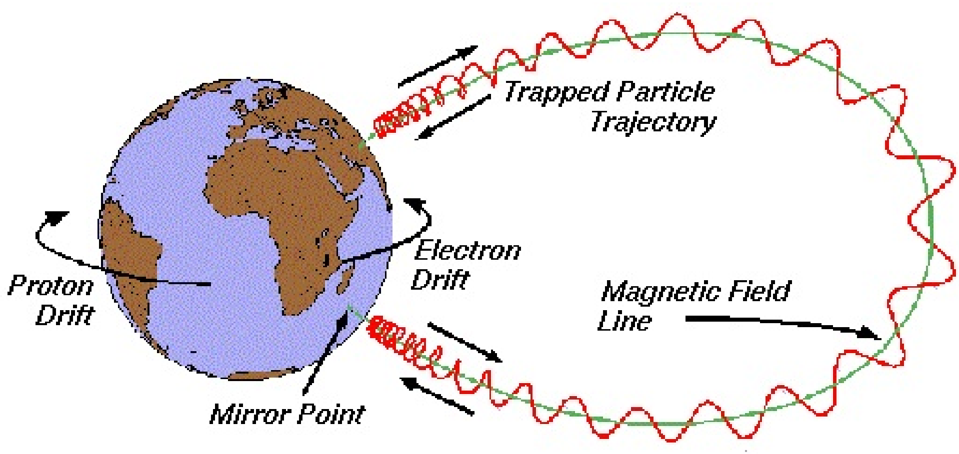


Figure 4.2: *Drifts and ring current in magnetosphere; image from NASA Earth ring current is an electric flows toroidally around the Earth. Azimuthally drifting particles trapped by the geomagnetic field create this current. Note that the changes in this current are responsible for global decreases in the Earth's surface magnetic field Daglis et al, 1999.*

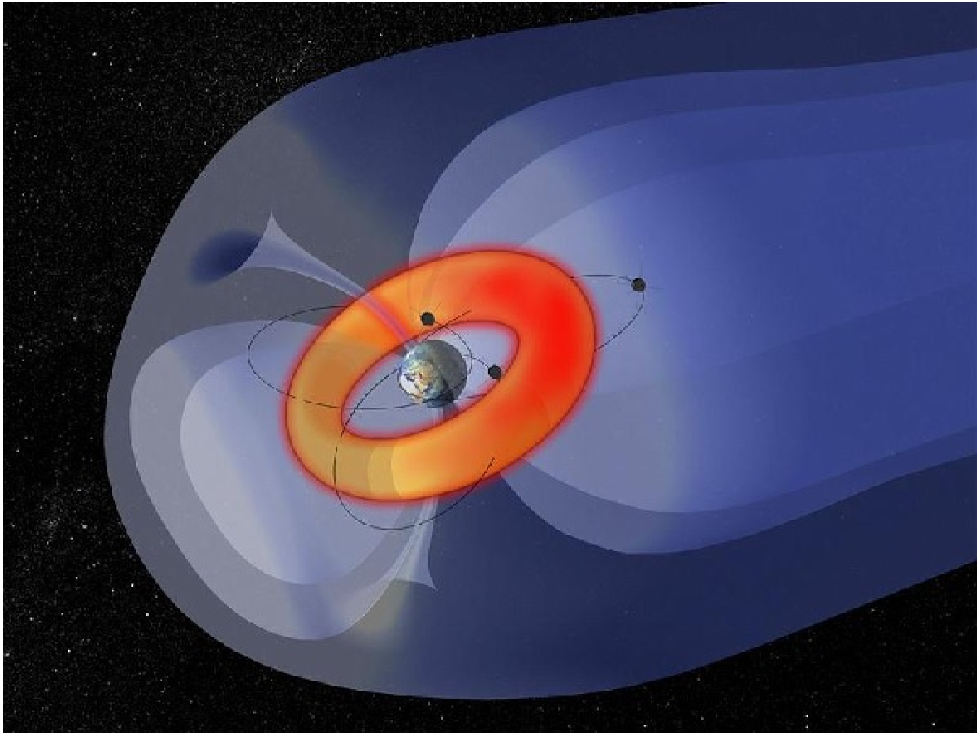


Figure 4.3: *Artist's impression of ring current from WAMI-project*

4.8 Importance of drifts

Drifts due to $\vec{\mathbf{B}}$ inhomogeneity is often exemplified via particle transport effects, e.g. ring current in the Earth magnetosphere (Figure 4.2).

However, ∇B and curvature **drifts are also important** in understanding particle acceleration at shock waves and current sheets (where $\vec{\mathbf{j}} \neq 0$).

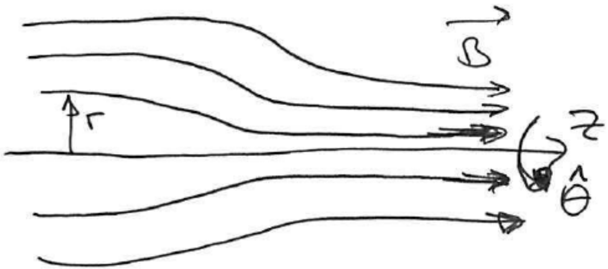
For example, consider the solar wind flow onto Earth bow shock (Fig 4.1), where magnetic field strength increases across the shock. The drifts along the solar wind convection electric field $-\vec{\mathbf{u}}_{sw} \times \vec{\mathbf{B}}_{sw}$ can lead to energy gain of the particles $q\vec{\mathbf{v}} \cdot \vec{\mathbf{E}} > 0$ (recall Equation 2.2). This mechanism is called **shock-drift acceleration** (for details see e.g. Bell & Melrose, 2001).

5 Magnetic mirroring, adiabatic invariants

LECTURE OUTLINE

- Magnetic mirroring, particle trapping and loss-cone
- Mirror force and magnetic moment
- Adiabatic invariants
- Particle trapping in lab devices and in astrophysics

5.1 Non-uniform magnetic fields



Consider non-uniform magnetic field with magnitude varying along z-direction, i.e. $\nabla B \parallel \vec{\mathbf{B}}$.

Let the field be axisymmetric, so that in cylindrical coordinates $B_\theta = 0$ and $\partial B/\partial\theta = 0$. Since the lines converge and diverge $B_r \neq 0$. From $\nabla \cdot \vec{\mathbf{B}} = 0$ we have:

$$\nabla \cdot \vec{\mathbf{B}} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

If $\dfrac{\partial B_z}{\partial z}$ at $r = 0$ is given and does not change much with r, we have:

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}dr \simeq -\frac{r^2}{2}\left.\frac{\partial B_z}{\partial z}\right|_{r=0}$$

The variation of B with r causes a grad-B drift of guiding centres about the axis of symmetry, but there is no radial grad-B drift, because $\partial B/\partial\theta = 0$. Hence the component of the Lorentz force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ are

$$q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = q\vec{\mathbf{e}}_r(\underbrace{v_\theta B_z}_{\text{gyration}} - 0) + q\vec{\mathbf{e}}_\theta(\underbrace{-v_r B_z}_{\text{gyration}} \underbrace{+v_z B_r}_{\text{azimuthal force}}) + q\vec{\mathbf{e}}_z(0 \underbrace{-v_\theta B_r}_{\text{force along } \vec{\mathbf{z}}})$$

$B_r \to 0$ vanishes at the axis $r \to 0$; when it does not vanish, this is azimuthal force leading to a drift in the radial direction. This drift makes the guiding centres follow the field lines.

Let us consider z-component

$$F_z = -qv_\theta B_r = +\frac{q}{2}v_\theta r\frac{\partial B_z}{\partial z}$$

and average over one gyro-orbit, setting $v_\theta = \mp v_\perp$, $r = r_L$ (see 1.12):

$$\langle F_z\rangle = \mp\frac{q}{2}v_\perp r_L\frac{\partial B_z}{\partial z} = -\frac{1}{2}\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z}$$

and finally

$$\langle F_z \rangle = -\frac{1}{2}\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z} \tag{5.1}$$

where we define the **magnetic moment**

$$\mu \equiv \frac{1}{2}\frac{mv_\perp^2}{B} \tag{5.2}$$

and the force can be written in terms of magnetic moment for the force acting parallel to magnetic field $\vec{\mathbf{B}}$:

$$\langle F_\parallel \rangle = -\mu\frac{\partial B}{\partial s} = -\mu\nabla_\parallel B \tag{5.3}$$

where ds is a line element directed along $\vec{\mathbf{B}}$. This force is known as **mirror force**.

5.2 Magnetic moment

The definition of **magnetic moment** μ is the same as the usual definition for magnetic moment of a current loop with area A and current I:

$$\mu = IA$$

The current $I = |q|\omega_c/2\pi$, area is $A = \pi r_L^2 = \pi v_\perp^2/\omega_c^2$, hence

$$\mu = IA = \frac{|q|\omega_c}{2\pi}\frac{\pi v_\perp^2}{\omega_c^2} = \frac{|q|m v_\perp^2}{2B|q|} = \frac{m v_\perp^2}{2B}$$

The current I is due to gyro-motion of a charged particle.

5.3 Magnetic moment conservation

Consider the component of the motion equation along $\vec{\mathbf{B}}$ when the magnetic field has $\nabla B \parallel B$ (from Equation 5.3):

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{\partial B}{\partial s}$$

Multiplying by $v_{\parallel}$:

$$v_{\parallel} m\frac{dv_{\parallel}}{dt} = \frac{d}{dt}\left(\frac{mv_{\parallel}^2}{2}\right) = -\mu\frac{\partial B}{\partial s}v_{\parallel} = -\mu\frac{\partial B}{\partial s}\frac{ds}{dt} = -\mu\frac{dB}{dt}$$

here dB is the variation of B as seen by the moving particle; B itself is constant.

Since $mv^2/2 =$ constant due to conservation of kinetic energy and from the definition of μ, $mv_{\perp}^2/2 = \mu B$, we have

$$\frac{d}{dt}\frac{mv^2}{2} = \frac{d}{dt}\left(\frac{mv_{\perp}^2}{2} + \frac{mv_{\parallel}^2}{2}\right) = \frac{d}{dt}\left(\frac{mv_{\parallel}^2}{2} + \mu B\right)$$

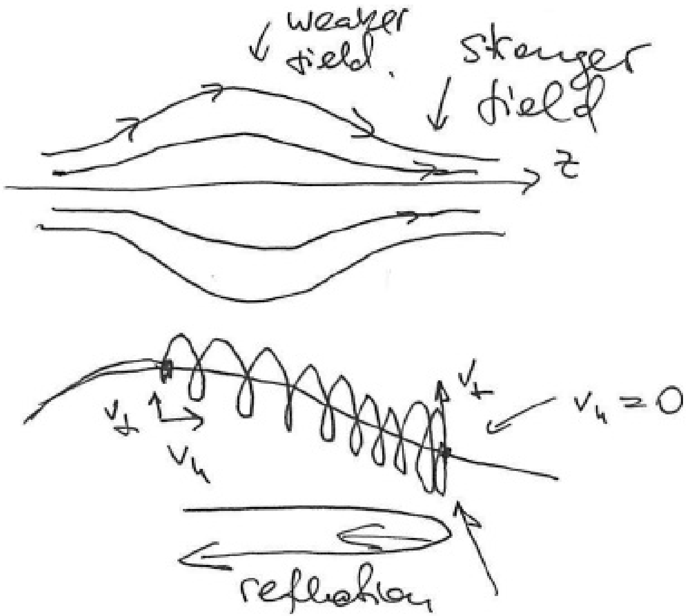
Combining with the previous equation

$$\frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = 0 \quad \Rightarrow \quad \frac{d\mu}{dt} = 0 \tag{5.4}$$

hence μ **is a constant of motion or an invariant**.

The invariance of μ is the basis for plasma/particle confinement using **magnetic mirrors**.

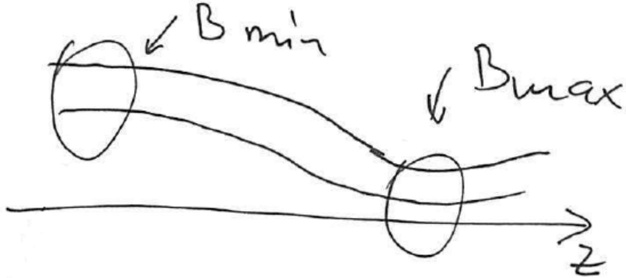
5.4 Magnetic trapping and mirroring



As the particle moves from weak field region to a strong field region, $v_\perp$ must increase to keep μ = const. Since $v_\perp^2 + v_\parallel^2$ = const, $v_\parallel$ decreases and eventually becomes zero. This point is called mirror point: the particle reflects back to the region of weaker field. This is due to the **mirror force**.

The plasma is trapped between magnetic mirrors, i.e. **magnetic trapping**. Note that this works both for electrons and ions.

5.5 Loss-cone



The trapping is not perfect, i.e. not for all particles. If a particle has $v_\perp = 0$, it will have $\mu = 0$ and hence will not feel the magnetic mirror force. So for small $v_\perp/v_\parallel$, we expect escape of the particles from the trap.

Consider a particle with $v_{\parallel 0}$ and $v_{\perp 0}$ in the region B_{min} initially.

$$\text{conservation of } \mu: \qquad \frac{1}{2}\frac{mv_{\perp 0}^2}{B_{min}} = \frac{1}{2}\frac{mv_\perp'^2}{B_{max}}$$

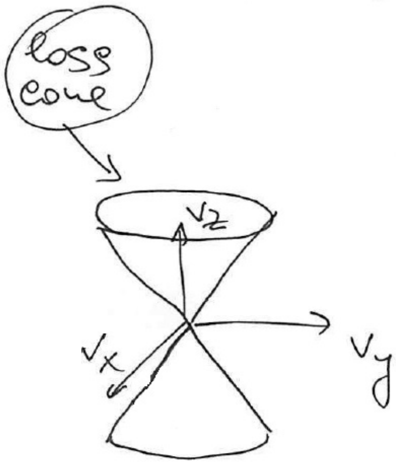
$$\text{conservation of } v^2: \qquad v_0^2 = v_{\perp 0}^2 + v_{\parallel 0}^2 = v_\perp'^2 + v_\parallel'^2$$

where $v_\parallel' = 0$ and $v_\perp'$ are at the mirror point.

Using these conservations, one finds that

$$\frac{B_{min}}{B_{max}} = \frac{v_{\perp 0}^2}{v_{\perp}'^2} = \frac{v_{\perp 0}^2}{v_0^2} = \sin^2\theta\,,$$

where $\sin\theta \equiv v_{\perp 0}/v_0$, θ is the **pitch angle of the particle**. Note that it is the angle in velocity space.



Depending on the initial $\sin\theta$, the particles can be either trapped or escape. When $\sin\theta_0$ of the particle in the region B_{min} such that

$$\sin^2\theta_0 < \sin^2\theta = \frac{B_{min}}{B_{max}} \quad \Rightarrow \quad \textbf{escape}$$

and the angle θ is **loss-cone** angle.

In the opposite case:

$$\sin^2\theta_0 > \sin^2\theta = \frac{B_{min}}{B_{max}} \quad \Rightarrow \quad \textbf{trapping}$$

5.6 Mirror devices

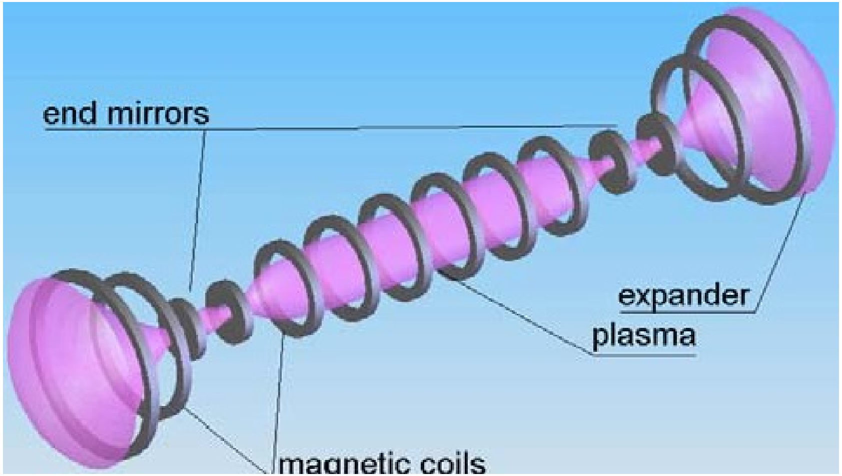


Figure 5.1: *magnetic mirroring occurs in nature as well as in plasma devices. A Magnetic Mirror Concept from Simonen et al, 2008*

5.7 The Earth's magnetic field and Van Allen radiation belts

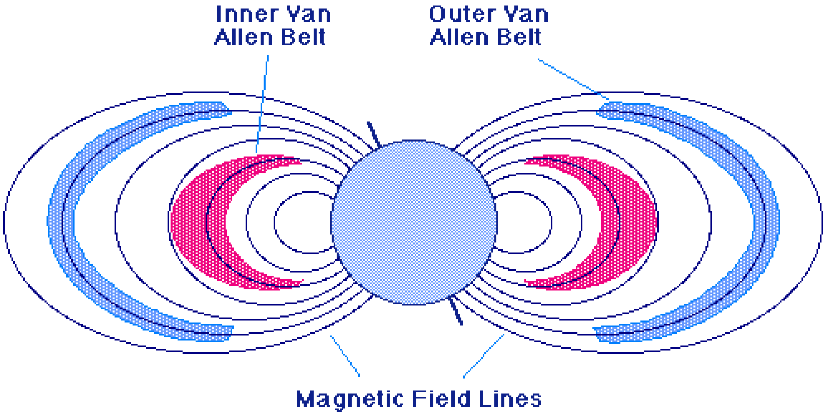


Figure 5.2: *Van Allen radiation belts consist of energetic particles trapped in the Earth's dipole-like magnetic field.* ***The inner belt****, 1-3 Earth radii is mostly populated by protons with energies exceeding* 10 *MeV. These protons are thought to be the decay of neutrons which are emitted from the Earth's atmosphere as it is bombarded by cosmic rays.* ***The outer belt****, about 3-9 Earth radii is mostly electrons with energies below* 10 *MeV. The origin of these electrons is via injection from the outer magnetosphere. See utexas.edu Figure from utk.edu*

5.8 Solar (and stellar) magnetic loops

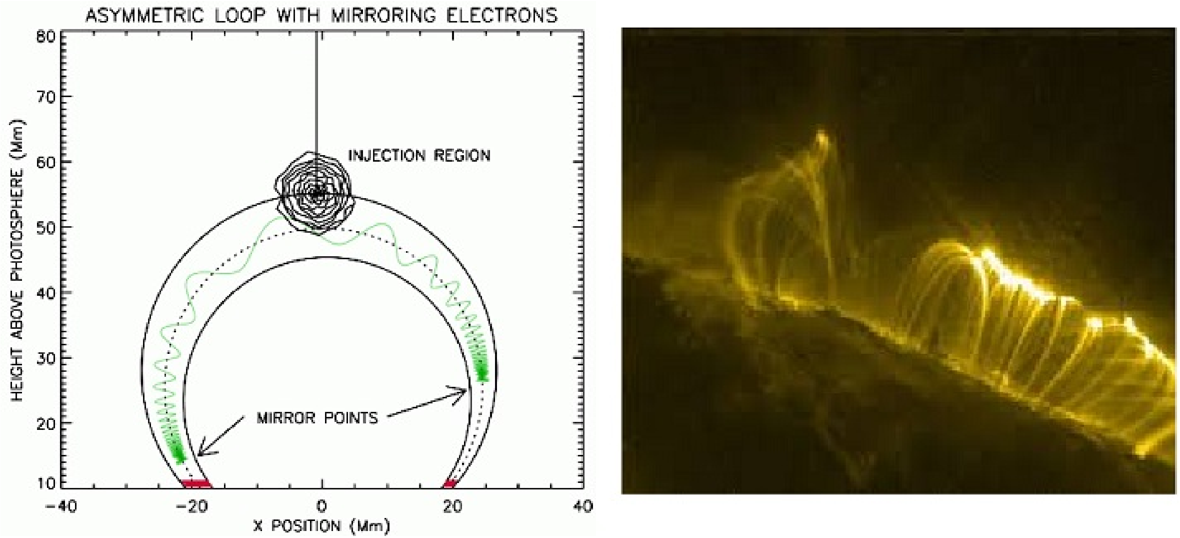


Figure 5.3: *Magnetic mirroring also could be important for particles in the* ***solar magnetic loops****. Left: Cartoon of trap/precipitation model for an asymmetric loop from by Ed Schmahl et al Right: EUV solar loops NASA SDO*

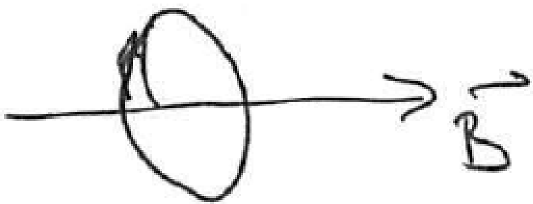
5.9 Adiabatic invariants

It is well-known in classical mechanics that whenever as system has a periodic motion, the action integral taken over a period is a constant:

$$\oint p dq = \text{const} \,,$$

where p and q are generalised momentum and coordinate, which repeat themselves in the motion. If a **slow change** is made in the system, so that the motion is not quite periodic, the constant of the motion does not change and is then called **adiabatic invariant**. The 'slow' here, slow compared with the period of motion, so the integral $\oint$ is well defined though it is strictly no longer an integral over a closed path.

5.10 First adiabatic invariant



Larmor gyration gives periodic motion (see the discussion of motion in constant B-field 1.12), so that

$$\oint pdq = \oint mv_\perp r_L d\phi = mv_\perp r_L 2\pi = 4\pi \frac{m}{|q|}\mu$$

so we find that $\mu = mv_\perp^2/2B$, magnetic moment is a constant. [See our derivation 5.4].

μ is conserved when $\vec{\mathbf{B}}$ changes slowly, e.g. Δt is much larger than gyro-period $\sim 1/\omega_c$

$$\Delta t \gg 1/\omega_c$$

when $\Delta t\omega_c < 1$, μ is **not** conserved.

5.11 Violation of the first adiabatic invariant

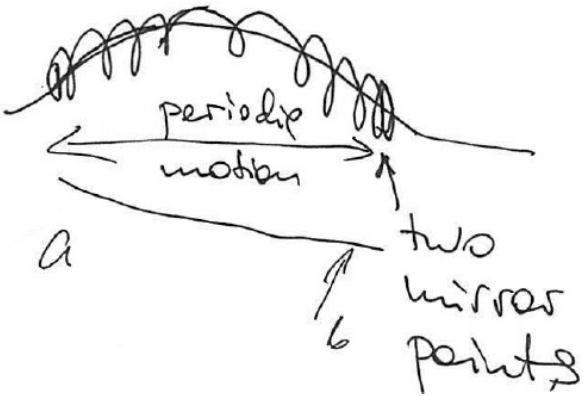
One of the most important examples of the violation is the presence of waves. The interaction is particularly strong when the wave frequency ω_w is the integer

$$\omega_w = n\omega_c, \quad n = 1, 2, 3, \ldots$$

which is called **cyclotron resonance**. Here, the rotating field of the wave leads to acceleration of a particle, increase of $v_\perp$ and violation of the first adiabatic invariant.

Similarly, spatial variation of B at $\perp$ distances $\sim r_L$ leads to violation of the invariant. Abrupt changes of magnetic field could appear in **shocks** and the other types of discontinuities. These effects are particularly important for ions, as they have larger larmor radii.

5.12 Second adiabatic invariant



When a particle bounces between two mirrors, there is a periodic motion at the **'bounce frequency'**. A constant of this motion is given by $\oint mv_{\|}ds$ where the integration is over the bounce period, s is the coordinate along $\vec{\mathbf{B}}$. Since guiding centre drifts across-field lines, the motion not exactly periodic and the constant of motion becomes **adiabatic invariant**. This is also called the **longitudinal invariant**, J

$$J = \int_a^b v_{\|}ds$$

and is defined for a half-cycle between mirror points a and b.

5.13 Violation of the second adiabatic invariant

One of the important examples of the violation of J invariance is given via by plasma heating (particle acceleration) scheme called **'transit-time magnetic pumping'**. This is achieved by creating a and b dependent on time, so that the particles see approaching mirror points. This leads to increase of $v_{\|}$.

The acceleration of particles by collisions against moving magnetic fields (suggested by Fermi, 1949) plays an important role in many astrophysical applications, e.g. cosmic rays, solar flares.

6 Radiation from charged particles

LECTURE OUTLINE

- Radiation by accelerated charged particle
- Larmor formula derivation
- Cyclotron and synchrotron radiation

6.1 Radiation from an accelerated charge

Charged particles can radiate if they move with acceleration.

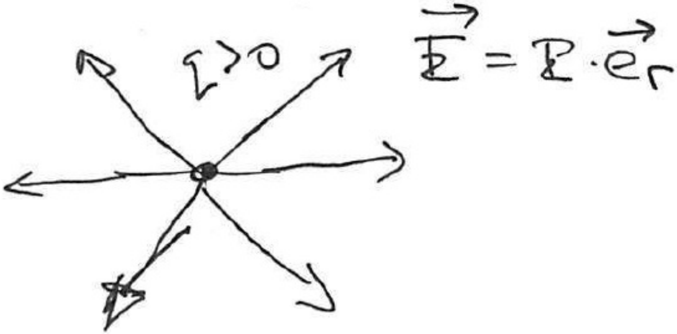
Gyromotion in magnetic field is an example of accelerated motion. Radiation from particles spiralling in a magnetic field is generally called **cyclotron radiation**. If radiating particles are relativistic, the radiation is **synchrotron radiation**.

Examples of such radiation: extragalactic jets, solar flares, supernova remnants, etc

Let us develop the expression for radiation from an accelerated charged particle, then apply it to the case of particles moving in helical orbits.

We will use simplified treatment (see *High Energy Astrophysics*, by Longair)

6.2 Charge field



Consider a charge q at $t = 0$, stationary at the origin of some laboratory rest frame. The electric field $\vec{\mathbf{E}}$ can be represented as radial lines from the charge:

$$\vec{\mathbf{E}} = \frac{q}{4\pi\varepsilon_0} \frac{\vec{\mathbf{r}}}{r^3} \qquad (6.1)$$

where $\vec{\mathbf{r}}$ is the radius vector centered at the charge.

Let the particle now accelerate to speed Δv in time Δt.

Here we assume that $\Delta v \ll c$, so that relativistic abberations are small.

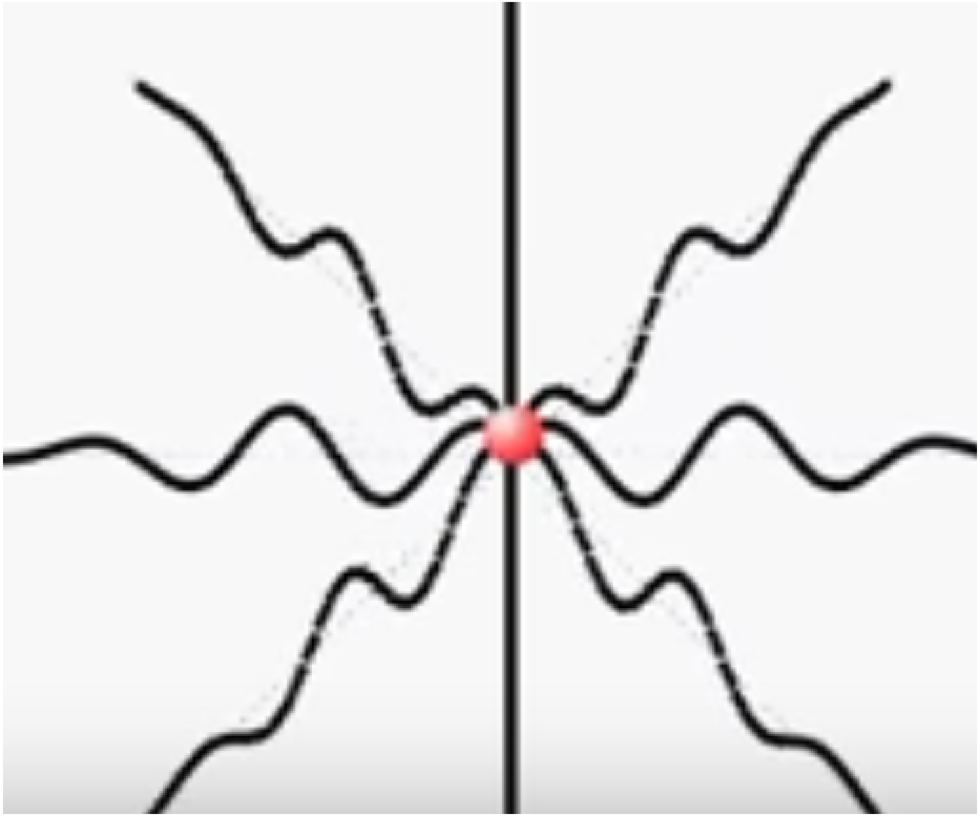
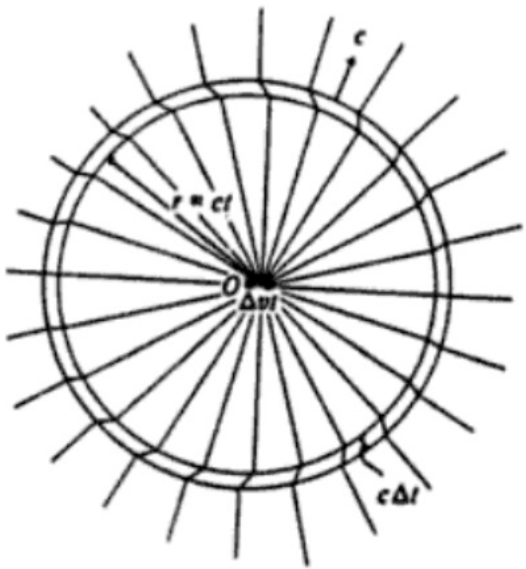


Figure 6.1: *Electric field of an oscillating charge. Animation available from* Youtube

6.3 The signal due to the distortion the electric field



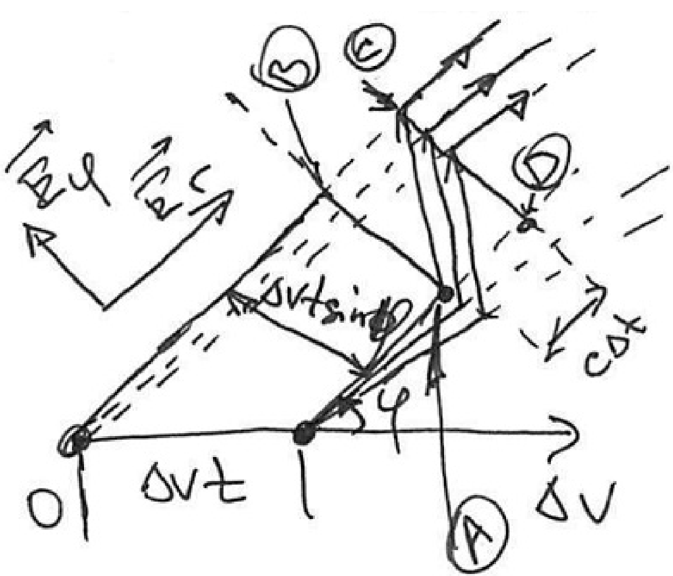
Within a distance $r = ct$, the field lines are radial, centered on the new position of the charge.

Outside $r = c(t + \Delta t)$, the field lines are also radial and have not responded to the change and still have their old configuration.

Between the two regions, there is a thin shell of thickness $c\Delta t$ across, which we have to join up the field lines.

Geometrically, there must be a **non-radial** component of $\vec{\mathbf{E}}$ in this small thin shell. This constitutes a propagating **'pulse'** of electromagnetic field.

6.4 Electric field in the pulse



Let us work out the electric field in the propagating pulse. Magnify up a small cone of the field. So the electric lines are radial at $t = 0$ and also at time t, but with different origins.

Join up these radial lines as shown. The field has radial component $\vec{\mathbf{E}}_r$ given by Equation (6.1) and $\vec{\mathbf{E}}_\varphi$ component. The strength of $\vec{\mathbf{E}}_\varphi$ component is given by the number of field lines per unit area in $\vec{\mathbf{e}}_\varphi$ direction. From geometry, the ratio of E_φ/E_r is given by the ratio of length AB to the length AD:

$$\frac{E_\varphi}{E_r} = \frac{\Delta v t \sin\varphi}{c\Delta t} \tag{6.2}$$

Substituting for E_r from (6.1):

$$E_\varphi = \frac{\Delta v t \sin\varphi}{c\Delta t}\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

where we set $t = r/c$ and $\Delta v/\Delta t = \dot{v}$ and obtain:

$$E_\varphi = \dot{v}\sin\varphi\frac{q}{4\pi\varepsilon_0}\frac{1}{c^2 r} \tag{6.3}$$

E_φ is known as the **'acceleration field'**. The strength varies as $1/r$ in contrast to the radial field of the charge which decreases as $1/r^2$.

This 'kink' in the electric field is an outward propagating pulse of electromagnetic radiation. Hence the energy per unit area per second at distance r into the direction $\vec{\mathbf{r}}$ is given by the Poynting vector:

$$\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} = c \underbrace{\epsilon_0 |\vec{\mathbf{E}}|^2}_{\text{energy density}} \frac{\vec{\mathbf{r}}}{r} \tag{6.4}$$

6.5 Power radiated

The total energy radiated per second P can be found by integrating $\vec{\mathbf{S}}$ over the surface of the sphere at distance r from the charge, so

$$P = \oint \vec{\mathbf{S}} d\vec{\mathbf{A}} = \oint_{\Omega} \vec{\mathbf{S}} r^2 d\vec{\mathbf{\Omega}}$$

and substituting expression for $\vec{\mathbf{S}}$ from 6.4, we find

$$P = \oint_{\Omega} \vec{\mathbf{S}} r^2 d\vec{\mathbf{\Omega}} = c\varepsilon_0 \int_0^{\pi} \left(\dot{v} \sin\varphi \frac{q}{4\pi\varepsilon_0} \frac{1}{c^2 r} \right)^2 r^2 \underbrace{2\pi \sin\varphi d\varphi}_{d\Omega} =$$

$$= \frac{q^2 \dot{v}^2}{8\pi\varepsilon_0 c^3} \underbrace{\int_0^{\pi} \sin^3\varphi d\varphi}_{=4/3}$$

Rewriting we derive **Larmor formula**

$$P = \frac{q^2 \dot{v}^2}{6\pi\varepsilon_0 c^3} \tag{6.5}$$

which gives the radiated power from an electric charge q.

The equation can be applied to e.g. the case of an electron gyrating in a magnetic field.

6.6 Cyclotron (synchrotron) radiation

Similar to **High Energy Astrophysics course** consider the rest frame of the charge q, i.e. the charge is stationary in this frame. Call this frame S'. The laboratory frame is S:

$$\text{S frame:} \qquad m\frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

$$\text{S' frame:} \qquad m\frac{d\vec{\mathbf{v}}'}{dt'} = q\vec{\mathbf{E}}'$$

Due to relativistic transformation of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, the electric field $E' = v\gamma B \sin\theta$, where γ is the Lorenz factor, θ is the particle pitch angle. Hence we have

$$\dot{v}' = \frac{q}{m}E'$$

Then using Larmor formula, we find

$$P' = \frac{q^2\dot{v}'^2}{6\pi\epsilon_0 c^3} = \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{qv\gamma B\sin(\theta)}{m}\right)^2$$

which is the power (energy per unit time) radiated in the rest frame of the electron. Since the power is Lorenz invariant $P' = P$.

In the lab frame, we find the power emitted by an particle with pitch angle θ

$$P = \frac{q^4}{6\pi\epsilon_0 c^3 m^2} v^2 \gamma^2 B^2 \sin^2(\theta) \tag{6.6}$$

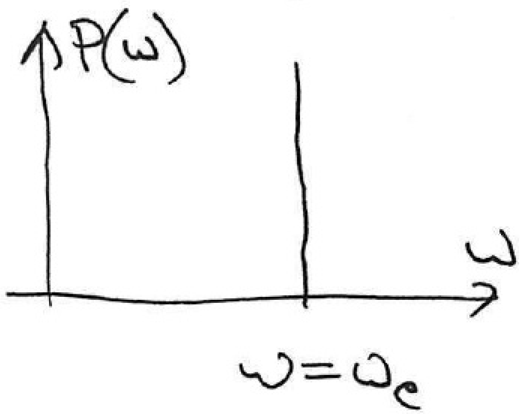
which can be re-written

$$P = 2\sigma_T c U_B \gamma^2 \sin^2\theta\,,$$

where σ_T is Thomson cross-section, $U_B = B^2/2\mu_0$ is the magnetic energy density[3].

[3]Recall High Energy Astrophysics I lectures

6.7 Spectrum of cyclotron and synchrotron radiation

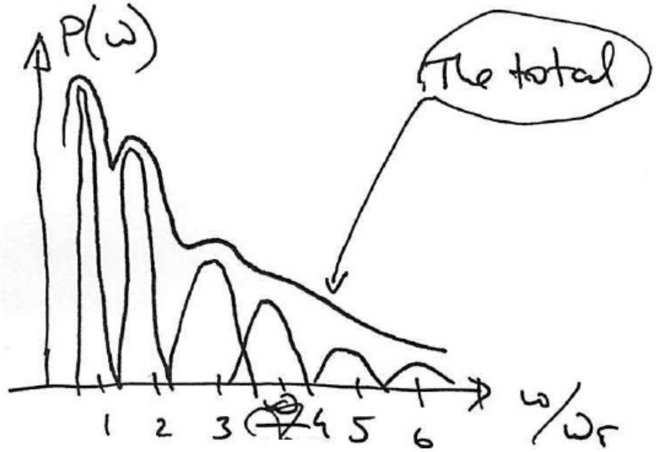


In the non-relativistic case, the radiation is emitted in a single line at the electron gyro-frequency.

In the mildly relativistic case, not all of the radiation is emitted at the gyro-frequency. The radiation spectrum can be decomposed by Fourier analysis into the sum of dipolar patterns, each radiating at integer multiplies **harmonics** of the relativistic gyro-frequency ω_r, where

$$\omega_r = \frac{\omega_c}{\gamma} = \frac{|q|B}{\gamma m_0},$$

where γ is the Lorentz factor, m_0 is the rest mass.



If the particle travelling at $v_\parallel$, with pitch angle θ, the harmonics are displaced because of the Doppler shift of the electron's radiation, so that n-th harmonic is at the frequency:

$$\omega_n = \frac{n\omega_r}{(1 - v_\parallel/c)}, \qquad n = 1, 2, 3, \ldots$$

The term $v_\parallel/c$ displaces the spectral lines from exact multiples of ω_r, which is the Doppler shift.

The lower curves show each harmonic component, and the upper lines gives the total $P(\omega)$.

6.8 Relativistic case (synchrotron radiation)

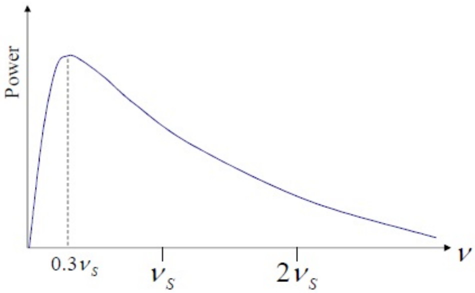


Figure 6.2: *Synchrotron spectrum of a single electron*

In the relativistic limit, as $v \to c$, the harmonics become higher and closer together. The spectrum of distinct peaks emitted by a single electron merges into a continuum [4].

Typical frequency of synchrotron radiation is

$$\nu_s = \frac{3}{2}\gamma^2 \left(\frac{eB}{2\pi m}\right) \tag{6.7}$$

Synchrotron radiation is emitted over a wide range of frequencies (Figure 6.2).

Peak occurs at $\sim 0.3\nu_s$, but **average** frequency value $\langle\nu\rangle \simeq \nu_s$. At low frequencies, $\nu \ll \nu_s$, the spectrum grows $\propto \nu^{1/3}$ (Figure 6.2).

[4] see HEAI lecture notes

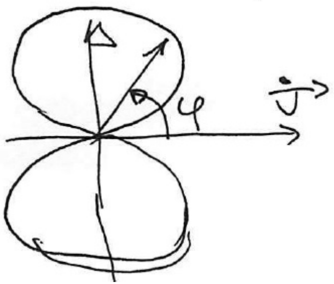
6.9 Relativistic beaming

Non-relativistic electron has a symmetric **dipolar radiation pattern in its rest frame**. Indeed, since E_φ depends on angle φ [Recall equation 6.3]

$$E_\varphi = \frac{q\dot{v}\sin\varphi}{4\pi\varepsilon_0 c^2 r}$$

then the power radiated into solid angle $d\Omega'$ in the particle rest frame is

$$\frac{dP'}{d\Omega'} = \frac{q^2\dot{v}^2\sin^2\varphi}{(4\pi)^2\varepsilon_0 c^3} \tag{6.8}$$



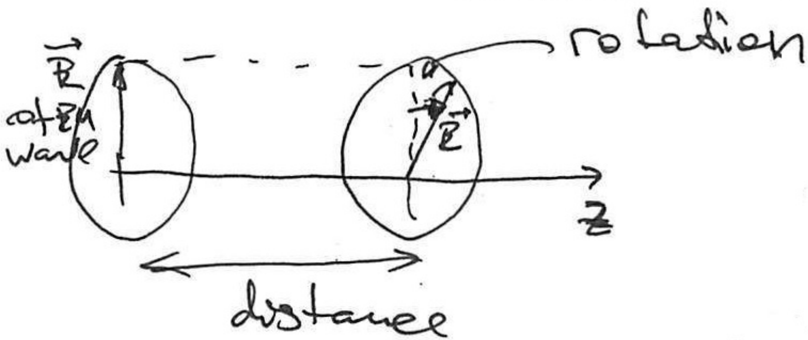
7 Faraday rotation

LECTURE OUTLINE

- Propagation of electromagnetic waves in magnetised plasma
- Faraday rotation angle

7.1 EM waves in plasma

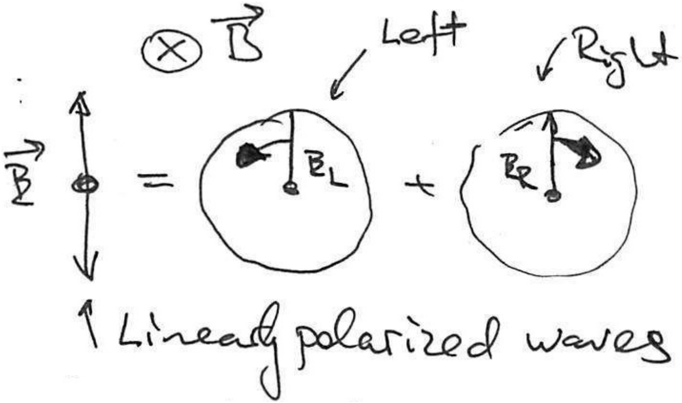
The radio waves (EM waves) generally has to pass through the plasma to reach the observer. As it propagates, EM wave **interacts** with the plasma



We will focus on the change of **polarization** as EM wave propagates through a magnetised plasma. This phenomenon (rotation of polarization vector) is called **Faraday rotation**.

The amount of rotation depends on both the magnetic field strength and the plasma density. Hence Faraday rotation can, in principle, be used to diagnose $\vec{\mathbf{B}}$ and n in plasma.

7.2 Left and Right CP polarized waves



Consider a linearly polarized wave as the sum of two counter-rotating **circularly** polarized waves - **left and right**.

$\vec{\mathbf{E}}$-field vector describes a circle as the wave propagates in z-direction. At a particular point in space, $\vec{\mathbf{E}}$ varies with time as

$$E_x = E_{xo} \cos \omega t$$
$$E_y = E_{yo} \sin \omega t$$

For a circularly polarized wave $|E_{xo}| = |E_{yo}|$, but in general $|E_{xo}| \neq |E_{yo}|$, i.e. elliptically polarized waves.

When **looking towards the source:**

$$\textbf{Right CP} \Longrightarrow \vec{\mathbf{E}} \quad \text{rotates anticlockwise}$$

$$\textbf{Left CP} \Longrightarrow \vec{\mathbf{E}} \quad \text{rotates clockwise}$$

Looking from the source RCP is LCP and LCP is RCP.

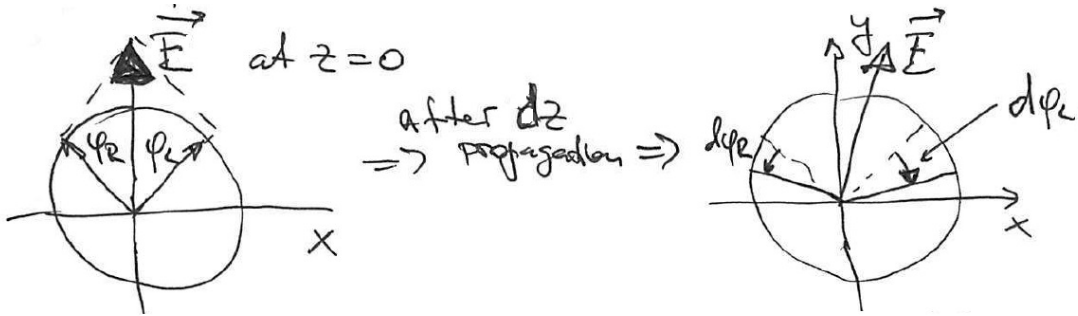


Figure 7.1: *Linearly-polarized wave can be considered as a superposition of two CP waves.*

As LCP and RCP waves prorogate differently through a magnetized plasma (Recall PTD course part1), so after some small distance dz, LCP and RCP

$\vec{\mathbf{E}}$-vectors will have rotated by small angles $d\varphi_L$ and $d\varphi_R$:

$$d\varphi_L = k_- dz$$
$$d\varphi_R = k_+ dz$$

where k_- and k_+ are the wave vectors of LCP and RCP waves respectively. As each wave advances in z, its polarization vector slowly rotates.

The net rotation of the vector superposition, i.e. the net rotation of the plane polarization is

$$d\varphi = \frac{1}{2}(d\varphi_L - d\varphi_R) = \frac{1}{2}(k_- - k_+)dz \tag{7.1}$$

Let us relate this change in polarization angle to the properties of the wave and the plasma in which it is travelling.

7.3 Dispersion relation

We can use the **dispersion relation** of the plasma, which relates the wavenumber, frequency and plasma parameters. For a cold plasma we have (see PTDI lecture notes) :

$$\begin{pmatrix} S - n^2\cos^2\theta & -iD & n^2\cos\theta\sin\theta \\ iD & S-n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P - n^2\sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{7.2}$$

where $n = kc/\omega$ is the refractive index, θ is the angle between $\vec{\mathbf{k}}$ and $\vec{\mathbf{B}}$, and the quantities S, D, and P are:

$$S = \frac{1}{2}(R+L) \qquad\qquad R = 1 - \frac{\omega_{pe}^2}{\omega(\omega-\omega_{ce})}$$
$$D = \frac{1}{2}(R-L) \qquad \text{where} \qquad L = 1 - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})}$$
$$P = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

7.4 High frequency waves

The dispersion relation (7.2) already includes the fact that $m_e/m_i \ll 1$ and we ignored ω_{ci} in comparison with ω_{ce}.

Let us consider EM wave with $\omega \gg \omega_{pe}$ and $\omega \gg \omega_{ce}$:

$$\begin{aligned} S =& \frac{1}{2}(R+L) = \frac{1}{2}\left(1 - \frac{\omega_{pe}^2}{\omega(\omega-\omega_{ce})} + 1 - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})}\right) \\ =& \frac{1}{2}\left(2 - \frac{\omega_{pe}^2}{\omega}\frac{(\omega+\omega_{ce})+(\omega-\omega_{ce})}{\omega^2-\omega_{ce}^2}\right) = 1 - \underbrace{\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}}_{=-r_p} = 1 + r_p \end{aligned}$$

so we can write[5]

$$
\begin{aligned}
S &= 1 + r_p \\
D &= r_p \frac{\omega_{ce}}{\omega} \\
P &= 1 - \frac{\omega_{pe}^2}{\omega^2} \underset{\omega \gg \omega_{ce}}{\underbrace{\simeq}} S
\end{aligned}
$$

[5] Check expression for D at home

7.5 Refractive index

We can consider a simplified version of refractive index, assuming propagating close to the direction of the field $\vec{\mathbf{B}}$, then

$$n_{\pm}^2 \simeq S \pm D \cos\theta \tag{7.3}$$

where $\pm$ corresponds to RCP and LCP waves respectively.

Substituting expressions for S and D, we have for $\omega \gg \omega_{pe}$ and $\omega \gg \omega_{ce}$:

$$n_{\pm}^2 \simeq 1 + r_p \pm r_p \frac{\omega_{ce}}{\omega} \cos\theta \simeq 1 - \frac{\omega_{pe}^2}{\omega^2} \mp \frac{\omega_{ce}\omega_{pe}^2}{\omega^3} \cos\theta$$

hence using Taylor expansion we have

$$n_{\pm} \simeq \left(1 - \frac{\omega_{pe}^2}{\omega^2} \mp \frac{\omega_{ce}\omega_{pe}^2}{\omega^3} \cos\theta\right)^{1/2} \simeq 1 - \frac{\omega_{pe}^2}{2\omega^2} \mp \frac{\omega_{ce}\omega_{pe}^2}{2\omega^3} \cos\theta \tag{7.4}$$

7.6 Rotation change

Using that $k = \omega n/c$ and the expressions for $n_\pm$ (Equation 7.4), we can now find $k_- - k_+$:

$$k_- - k_+ = \frac{\omega}{c}(n_- - n_+) = \frac{\omega}{c}\frac{\omega_{ce}\omega_{pe}^2}{\omega^3}\cos\theta$$

Hence we find the **Faraday rotation angle**:

$$d\varphi = \frac{1}{2}\frac{\omega_{ce}\omega_{pe}^2}{c\omega^2}\cos\theta dz \tag{7.5}$$

Left-handed circularly polarized wave will travel at a slightly lower phase velocity, than RCP increasing the polarization angle.

Using $\omega_{pe} = \sqrt{e^2 n_e / m\varepsilon_0}$ and $\omega_{ce} = eB/m$

$$\frac{d\varphi}{dz} = \frac{1}{2}\frac{e^2 n_e}{m\epsilon_0}\frac{eB}{m}\frac{\lambda^2}{(2\pi)^2 c^3}\cos\theta = \frac{e^3 B n_e \lambda^2}{8\pi^2 m^2 \epsilon_0 c^3}\cos\theta$$

To find the total Faraday rotation angle φ occuring over a path r, we integrate along the path

$$\varphi_F = \frac{e^3 \lambda^2}{8\pi^2 m^2 \epsilon_0 c^3}\int_0^r B(z) n_e(z)\cos\theta dz \qquad (7.6)$$

Assuming a uniform electron number density $n_e =$ const, φ_F is a measure of B along the line-of-sight to the observer.

The physical reason for the Faraday rotation is that the two signs of circular polarization interact differently with the plasma - the plasma has a different refractive index for different polarizations of EM waves. The **phase speed** for waves, ω/k with polarization vector rotating with the direction of spiraling of electrons in somewhat higher that that a wave with polarisation vector rotating in the opposite direction to electron spiralling.

The rotation is higher for longer wavelength waves $\varphi_F \propto \lambda^2$ or lower frequency waves. Hence important for radio observations.

7.7 Faraday rotation measurements

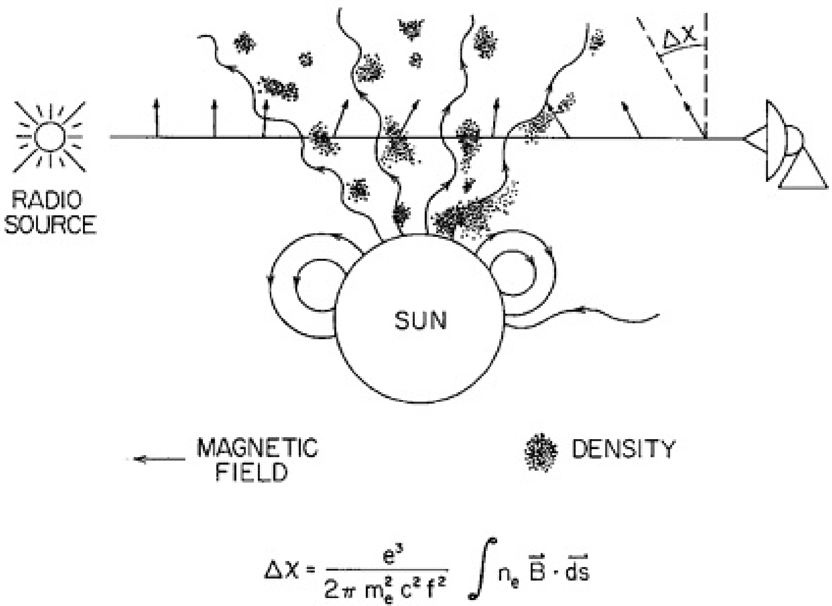


Figure 7.2: *Illustration of coronal faraday rotation from Spangler, 2005*

Faraday rotation is used to measure B (making assumption on plasma density) by looking through this plasma, e.g. in the interstellar medium, in the solar corona, in our Galaxy (by observing large number of sources)

7.8 Rotation measure maps

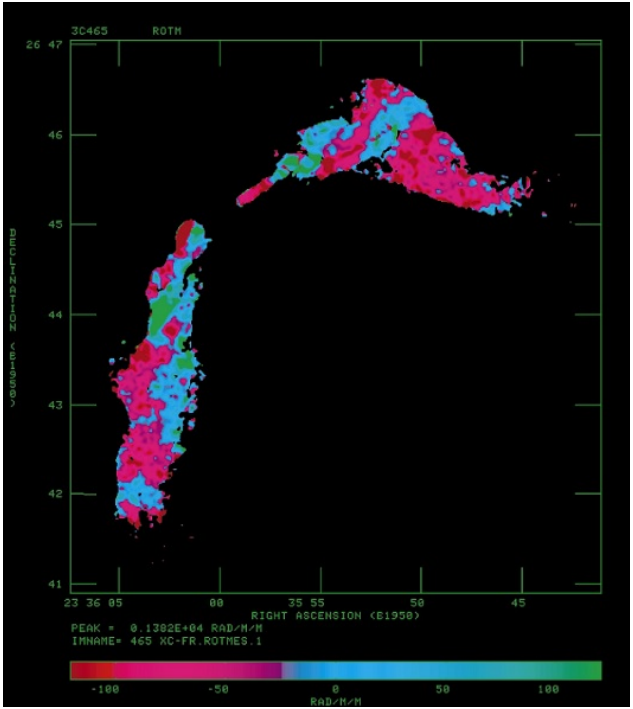


Figure 7.3: *Rotation measure distribution in galaxy cluster Abell 2634 from Eilek & Owen, 2002*

8 Coulomb collisions

LECTURE OUTLINE

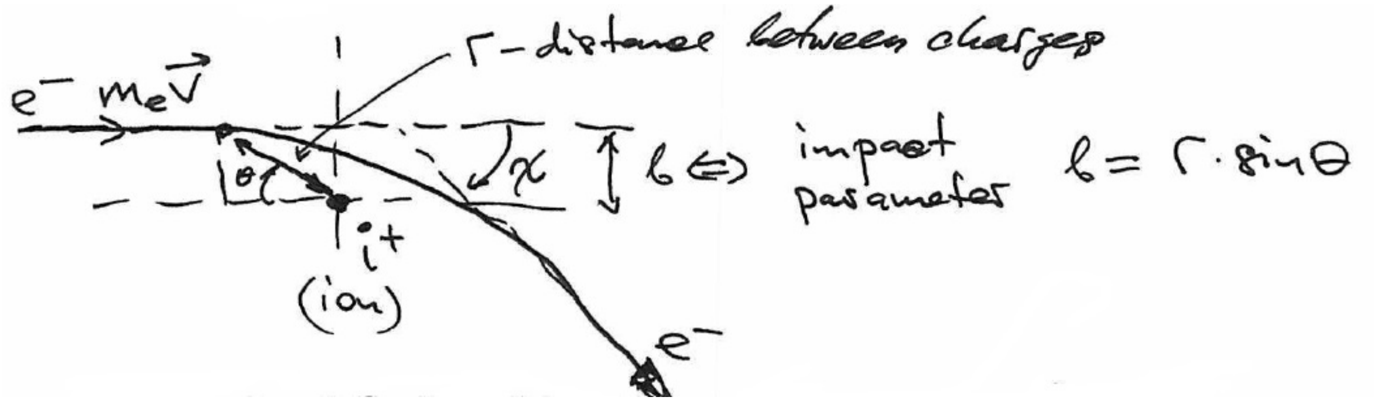
- Coulomb collisions in plasma
- Coulomb logarithm
- Mean free path
- Electron-ion, electron-electron and ion-ion collisions

8.1 Binary Coulomb collision

Let consider a **binary Coulomb collision** of two plasma particles (e.g. $e-e$, $i-i$, or $e-i$). The Coulomb force between tow charges is

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \tag{8.1}$$

where q_1,q_2 are the charges, r is the distance between the charges. For simplicity, consider electron interacting with a heavy ion ($m_i >> m_e$):



and introduce **impact parameter** b, and **scattering angle** χ.

8.2 Change of perpendicular velocity

Let us consider the change of $v_\perp$ when we have a massive (immobile) ion $m_i \gg m_e$. The change of **perpendicular momentum** from equation of motion $m_e d\vec{\mathbf{v}}/dt = \vec{\mathbf{F}}$ can be written:

$$m_e v_\perp = \int_{-\infty}^{\infty} F dt \simeq F\Delta t$$

where Δt is the time of interaction. Let approximate:

$$m_e v_\perp \simeq F\Delta t \simeq \frac{q_e q_i}{4\pi\epsilon_0 r^2}\frac{r}{v} = \frac{q_e q_i}{4\pi\epsilon_0 r v}$$

and $r \simeq b$.

For large-angle collisions, i.e. $\chi \sim 90^o$, the change of $m_e v_\perp$ is of the order of mv itself, so

$$m_e v_\perp \simeq m_e v \simeq \frac{q_e q_i}{4\pi\epsilon_0 b v}$$

so we can estimate **impact parameter** b:

$$b = \frac{q_e q_i}{4\pi\epsilon_0 m_e v^2} \tag{8.2}$$

8.3 Collision cross-section and collision frequency

The interaction between two particles can be described using the **cross-section of the interaction**, σ, [units of area] [6] which we take as the area of the disk with radius b:

$$\sigma = \pi b^2$$

We simply say that the interaction is happening for impact parameters less than b from Equation 8.2 and no interaction for larger b. Then the cross-section of interaction can be written:

$$\sigma_{ei} = \pi \frac{q_e^2 q_i^2}{(4\pi\epsilon_0)^2 (m_e v^2)^2} \tag{8.3}$$

and the **collision frequency** or collision rate:

$$\nu_{ei} = n\sigma_{ei} v \tag{8.4}$$

[6] recall High Energy Astrophysics I, see HEAI notes on moodle

Then $e - i$ collision frequency for $z = 1$ plasma (hydrogen plasma), with $n_e = n_i = n$ and $q_e = -q_i = e$, where e is the electron charge, becomes:

$$\nu_{ei} = n\sigma_{ei}\upsilon \simeq \frac{\pi e^4 n\upsilon}{(4\pi\epsilon_0)^2(m_e\upsilon^2)^2} \propto \frac{n}{\upsilon^3}$$

For the electrons with thermal energy $k_B T_e = m_e \upsilon_{Te}^2/2$, $\upsilon_{Te} = (2k_B T_e/m_e)^{1/2}$, so we find

$$\nu_{ei} \simeq \frac{\sqrt{2}}{64\pi}\frac{\omega_{pe}^4}{n}\left(\frac{k_B T}{m}\right)^{-3/2} = \frac{\pi n e^4}{2^{3/2}(4\pi\epsilon_0)^2 m^2 (k_B T_e/m)^{3/2}} \propto n T_e^{-3/2}$$

Note that **this is a rough estimate**, there are many small-angle (small χ) collisions in plasma.

8.4 Coulomb logarithm

A more rigorous estimate gives (see book by Francis F. Chen (2006) [GU Library link])

$$\nu_{ei} = \frac{4\sqrt{2}}{3\sqrt{\pi}} \frac{\pi n e^4 \ln \Lambda}{(4\pi\epsilon_0)^2 m_e^{1/2} (k_B T_e)^{3/2}} \tag{8.5}$$

where $\Lambda \simeq n_e \lambda_{De}^3$ is the number of particles in Debye sphere and $\lambda_{De} = (\epsilon_0 k_B T_e / e^2 n)^{1/2}$ is Debye length, where $\ln \Lambda$ is called **Coulomb logarithm**, and normally assumed to be a constant number $\ln \Lambda \simeq 10 - 20$ in astrophysical plasmas.

8.5 Mean free path

Let us estimate the **mean free path** of an electron in plasma

$$\lambda = \frac{v}{\nu_{ei}} \simeq \frac{v_{Te}}{\nu_{ei}} \simeq \frac{\omega_{pe}\lambda_{De}}{\nu_{ei}} \simeq \left(\frac{\omega_{pe}}{\nu_{ei}}\right)\lambda_{De}$$

where normally $\omega_{pe}/\nu_{ei} \gg 1$.

Substituting constants into Equation (8.5), we can write

$$\nu_{ei} \simeq \frac{5 \times 10^{-11} n_e[m^{-3}]}{(k_B T_e[eV])^{3/2}} \left(\frac{\ln \Lambda}{17}\right), \qquad [\mathrm{s}^{-1}] \tag{8.6}$$

For, say solar corona plasma $n \sim 10^{15}$ m^{-3}, $k_B T_e \sim 10^2$ eV, we estimate, $\nu_{ei} \simeq 50$, [s^{-1}].

Using that $\nu_{pe} = \omega_{pe}/2\pi \simeq 9 \times (n_e[\mathrm{cm}^{-3}])^{1/2}$ [kHz], we estimate $\nu_{pe} \sim 3 \times 10^8$ [s^{-1}], which is much larger than ν_{ei}. Hence the plasma can be viewed **collisionless**. Hence **the collisional mean free path is much larger than the Debye length**.

8.6 Multiple collisions and mean free path

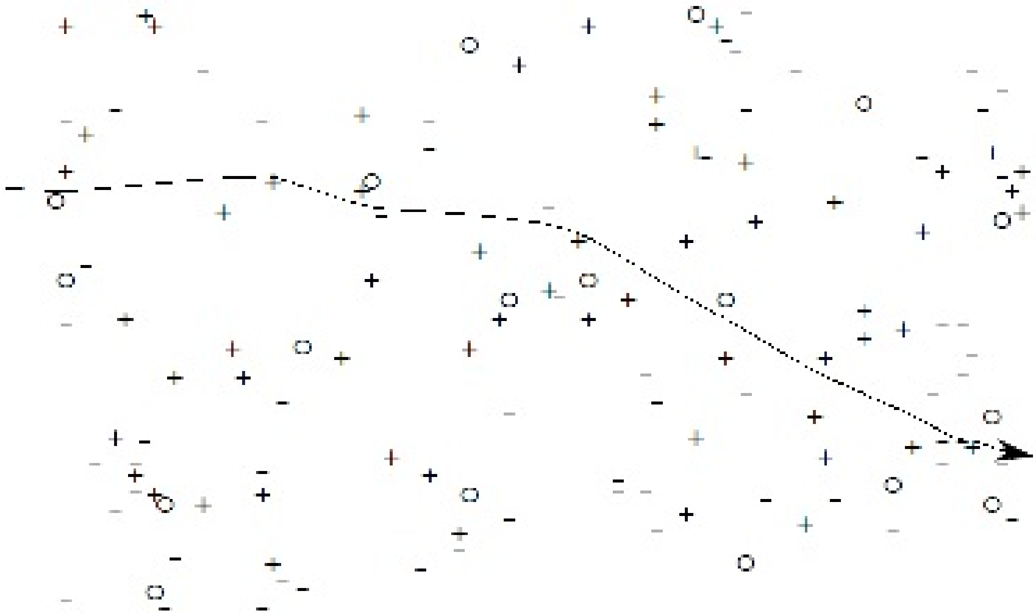


Figure 8.1: *A charge travelling in plasma experiences many small-angle scatterings. The mean collision length of a charged particle is the* ***average distance*** *it moves in being deflected so that $\Delta \upsilon \simeq \upsilon$, from Cullen 2006.*

8.7 Electron-ion, electron-electron, and ion-ion equilibration

Consider plasma with $T_e \sim T_i$ and $n_e \simeq n_i$. For electron-ion collisions in plasma we found $\nu_{ei} \propto n/(m_e^{1/2} T_e^{3/2})$ the quantity $\tau_{ei} \equiv 1/\nu_{ei}$ is the **average time between $e - i$ collisions**, or mean free time.

For $e - e$ collision frequency (we need to take into account finite mass of the scattering particle and replace it with m_e), this gives a factor of 2, so

$$\nu_{ee} \simeq \nu_{ei}/\sqrt{2}$$

For $i - i$ collisions m_e to be changed to m_i in Equation (8.5), so we have

$$\nu_{ii} \simeq \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{ee}$$

For '**ion-electron** collisions' (centre of mass transformation gives a factor m_e/m_i), hence we have:

$$\nu_{ie} \simeq \frac{m_e}{m_i} \nu_{ee}$$

These times are **momentum** loss times.

8.8 Collisional energy exchange

Let us consider now $T_e \neq T_i$, and find the time of energy exchange between $e-e$, $e-i$ and $e-i$. The **energy exchange** times are

$$\tau_{ee}^{E} : \tau_{ii}^{E} : \tau_{ei}^{E} \sim 1 : \left(\frac{m_i}{m_e}\right)^{1/2} : \frac{m_i}{m_e} \tag{8.7}$$

we note ν_{ei} and $\tau_{ei} = \nu_{ei}^{-1}$ is not the (time) of the establishment of thermal equilibrium between the electrons and ions; it describes the rate of momentum transfer from electrons to the ions, not the rate of energy exchange between them. The relaxation time for electrons-ion eqvilibrium is given by ion-electron collision and $\sim m_e/m_i$ slower than ν_{ee}.

Hence for Hydrogen plasma $m_i/m_e = 1836$, we have:

$$\tau_{ee} : \tau_{ii} : \tau_{ei}^{E} \sim 1 : 43 : 1836$$

Electrons and ions equilibrate among themselves much faster than with each other. Different T_e and T_i are often observed in astrophysical plasmas (see book by Pitaevskii and Lifshitz [GU library]).

9 Collisional resistivity and diffusion in plasma

LECTURE OUTLINE

- Collisional diffusion of particles in plasma
- Collisional resistivity and electrical conductivity
- Driecer field

9.1 Collisional momentum and energy change

As we saw, collisions change momentum and energy of particles. It has important implications.

The **energy loss collision frequency**, which is to do with slowing down to rest and exchanging energy, is important for e.g. equilibration times (of temperatures) and energy transfer between species.

The **momentum loss frequency**, which is to do with loss of directed velocity, is required for calculating mobility: conductivity/resistivity, viscosity, particle diffusion, energy (thermal) diffusion.

9.2 Collisional resistivity and electrical conductivity

Consider unmagnetised, quasi-neutral $n_i \simeq n_e$ plasma of ions and electrons q_i = q_e = e. In response to an applied electric field $\vec{\mathbf{E}}$, an **electric current** will flow in plasma. The current density is:

$$\vec{\mathbf{j}} = n_i e \vec{\mathbf{v}}_i - n_e e \vec{\mathbf{v}}_e$$

Since electrons have $m_e \ll m_i$, so the plasma current is carried mostly by electrons. Hence consider electron momentum equation (**fluid**):

$$m_e n_e \frac{d\vec{\mathbf{v}}_e}{dt} = -e n_e \vec{\mathbf{E}} + \underbrace{m_e n_e \nu_{ei}(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_e)}_{\text{due to collisions}} \tag{9.1}$$

In steady state, we have $d\vec{\mathbf{v}}_e/dt = 0$, so we find

$$\vec{\mathbf{E}} = \frac{m_e \nu_{ei} \overbrace{n_e(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_e)}^{=\vec{\mathbf{j}}/e}}{e n_e} = \frac{m_e \nu_{ei}}{e^2 n_e} \vec{\mathbf{j}}$$

Ohm's law says $\vec{\mathbf{E}} = \eta\vec{\mathbf{j}}$, so we derive the resistivity due to collisions:

Spitzer resistivity: $$\eta = \frac{m_e \nu_{ei}}{e^2 n_e} \tag{9.2}$$

or **classical electrical conductivity** $\sigma = 1/\eta$:

electrical conductivity: $$\sigma = \frac{e^2 n_e}{m_e \nu_{ei}} \tag{9.3}$$

Since $\nu_{ei} \propto n_e T_e^{-3/2}$, the collisional resistivity

$$\eta = \frac{m_e \nu_{ei}}{e^2 n_e} \propto \frac{\cancel{n_e}}{\cancel{n_e}} T_e^{-3/2}$$

is **independent** of number of charge carriers, n_e and decreases with growing temperature.

Note that because of the conservation of the total electron momentum, $e - e$ collisions do not contribute to the resistivity.

9.3 Plasma resistivity in Astrophysics

Collisions of electrons with ions (or other species) in the plasma lead to resistivity and provide a mechanism for heating. This mechanism is often called **ohmic heating or Joule heating**.

Plasma electric conductivity is usually very high: For many purposes, the conductivity of a plasma may be treated as infinite. This leads to frozen-in condition.

9.4 Driecer field

Electric fields parallel to magnetic fields can accelerate charged particles. collisions change momentum and energy of particles. So friction-like term should appear in equations of motion. Consider 1D equation of electron motion:

$$m\frac{dv}{dt} = eE - \nu_{ei}mv$$

There is a critical velocity that sets right hand side to zero. Electrons with the velocities larger than the critical are accelerated. The process is called **electron runaway**.

Assuming thermal distribution of electrons, and $v = v_{Te}$, there is critical electric field, called **Dreicer field** (Dreicer, 1959):

$$E_D = \frac{\nu_{ei}(v = v_{Te})mv_{Te}}{e} = \frac{e^3 n \ln \Lambda}{6\pi\epsilon_0^2 k_B T} \tag{9.4}$$

Note that the resistivity (9.2) is valid for $E \ll E_D$.

9.5 Diffusion of particles

The fluid equation of motion including collisions, say, for electrons:

$$m_e n \frac{d\vec{\mathbf{v}}}{dt} = qn\vec{\mathbf{E}} - \nabla p - m_e n \nu_{ei} \vec{\mathbf{v}}$$

where again we assume $n_i = n_e = n$. We will also assume that ν_{ei} is a constant. We shall consider steady state, so that $d\vec{\mathbf{v}}_e/dt = 0$

$$qn\vec{\mathbf{E}} - \nabla p - m_e n \nu_{ei} \vec{\mathbf{v}} = 0$$

Using $p = nk_B T$ and assuming isothermal plasma

$$qn\vec{\mathbf{E}} - k_B T \nabla n - m_e n \nu_{ei} \vec{\mathbf{v}} = 0$$

so the fluid velocity $\vec{\mathbf{v}}$ becomes:

$$\vec{\mathbf{v}} = \frac{qn\vec{\mathbf{E}} - k_B T \nabla n}{m_e n \nu_{ei}} = \underbrace{\frac{q}{m_e \nu_{ei}}}_{\text{mobility}} \vec{\mathbf{E}} - \underbrace{\frac{k_B T}{m_e \nu_{ei}}}_{\text{diffusion}} \frac{\nabla n}{n}$$

where $\mu = |q|/m_e\nu_{ei}$ and $D = k_B T/m_e\nu_{ei}$ are mobility and diffusion coefficients. These will be different for each species, D is measured in [m^2/sec]. We can write the flux $\vec{\mathbf{\Gamma}}$:

$$\vec{\mathbf{\Gamma}} \equiv n\vec{\mathbf{v}} = \pm\mu n\vec{\mathbf{E}} - D\nabla n$$

in case $\vec{\mathbf{E}} = 0$, or particles are not charged we find **Fick's law**:

$$\vec{\mathbf{\Gamma}} = -D\nabla n \tag{9.5}$$

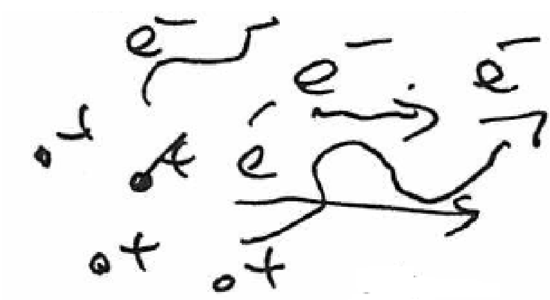
Using continuity equation $\partial n/\partial t + \nabla \cdot \vec{\mathbf{\Gamma}} = 0$, we can find equation

$$\frac{\partial n}{\partial t} - \nabla D\nabla n = 0, \tag{9.6}$$

which is the diffusion equation.

Hence, the particles **diffuse in space** because of the collisional scattering.

9.6 Ambipolar diffusion



Plasma should be quasi-neutral, so the diffusion of electrons and ions should adjust to preserve quasi-neutrality. The fast electrons have higher thermal velocities and tend to leave plasma first. Positive charge is left and an electric field is set up as to retard the loss of electrons and accelerate the loss of ions. So we set:

$$\vec{\mathbf{\Gamma}}_e = \vec{\mathbf{\Gamma}}_i = \vec{\mathbf{\Gamma}}$$

or

$$\vec{\mathbf{\Gamma}} = +\mu_i n\vec{\mathbf{E}} - D_i \nabla n = -\mu_e n\vec{\mathbf{E}} - D_e \nabla n$$

solving for $\vec{\mathbf{E}}$, we find:

$$\vec{\mathbf{E}} = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}$$

so the total flux $\vec{\boldsymbol{\Gamma}}$ is

$$\vec{\boldsymbol{\Gamma}} = \mu_i n \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\nabla n}{n} - D_i \nabla n = -\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} \nabla n$$

So similar to (9.5), we have Fick's law, but with a new coefficient

$$D_A = \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} \tag{9.7}$$

called **ambipolar** diffusion coefficient, i.e. two kinds of particles tend to have a diffusion rate which is intermediate in value to their free diffusion rates.

9.7 Thermal conduction in plasma

Let us examine plasma with a temperature gradient, i.e. $\nabla T_e \neq 0$. If the plasma is collisional, we expect diffusion motion of electrons:

$$\vec{\mathbf{\Gamma}} = -D\nabla n \propto \frac{k_B T}{m_e \nu_{ei}} \nabla n \tag{9.8}$$

Recall that electric current in collisional plasmas is carried predominately by the fast electrons in the distribution function, so the energy flux is also primarily due to electrons. If more fast electron diffusively move in the direction, and more slow electrons move in the opposite direction to have zero net current, there will be net energy flux. The energy flux with **Spitzer thermal conductivity**, Spitzer, L., 1956)

$$\vec{\mathbf{Q}} \propto -\frac{n k_B T}{\nu_{ei}} \nabla T \propto T^{5/2} \nabla T \tag{9.9}$$

Note strong plasma temperature dependency and only ln n dependency via Coulomb Logarithm.

10 Kinetic description of plasmas

LECTURE OUTLINE

- Particle distribution function
- Kinetic equation
- Plasma waves in collisionless plasma

10.1 Particle distribution function

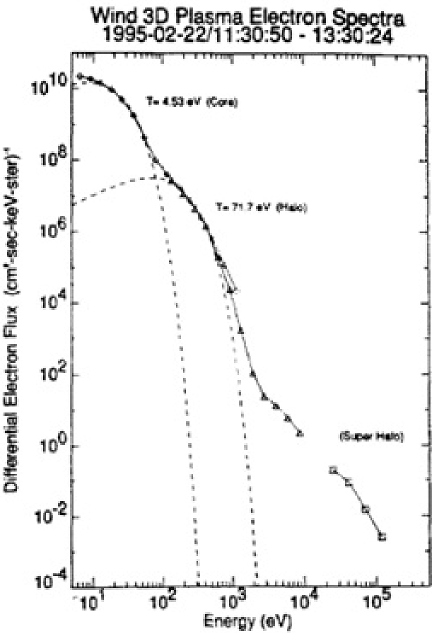


Figure 10.1: *Differential flux spectrum of electrons measured in the solar wind Lin et al 1997*

There are some phenomena, for which fluid (MHD) or single particle description are not adequate. For these we need to consider the velocity (or momentum in relativistic case) distribution function $f(\vec{\mathbf{v}})$, (e.g. Fig 10.1). This treatment is called **kinetic**.

In fluid theory, the dependent variable $\vec{\mathbf{u}}$, ρ, p, etc are functions of $\vec{\mathbf{r}}$ and t only. This is possible because $f(\vec{\mathbf{v}})$ is assumed to be Maxwellian everywhere and therefore can be uniquely specified by temperature T and number density

$$n(\vec{\mathbf{r}},t) = \int f(\vec{\mathbf{v}},\vec{\mathbf{r}},t)d^3v\,.$$

Alternatively, if f is normalised so that $\tilde{f}$

$$\int \tilde{f}(\vec{\mathbf{v}},\vec{\mathbf{r}},t)d^3v = 1, \qquad f = n(\vec{\mathbf{r}},t)\tilde{f}$$

then $\tilde{f}$ is the probability to find a particle in the range $(\vec{\mathbf{r}},\vec{\mathbf{r}}+d\vec{\mathbf{r}})$ and $(\vec{\mathbf{v}},\vec{\mathbf{v}}+d\vec{\mathbf{v}})$. The function f or $\tilde{f}$ is the function of 7 variables, $f(\vec{\mathbf{v}},\vec{\mathbf{r}},t)$ and units are [m^{-3} (m/s)$^{-3}$]. Maxwellian distribution is

$$f(\vec{\mathbf{v}}) = n(\vec{\mathbf{r}},t)\frac{1}{(2\pi k_B T/m)^{3/2}}\exp\left(-\frac{v^2}{v_T^2}\right), \tag{10.1}$$

where $v_T = (2k_BT/m)^{1/2}$, k_B is the Boltzmann constant.

Knowing $f(\vec{\mathbf{v}},\vec{\mathbf{r}},t)$, one can calculate n, T, etc. taking moments of the distribution function.

10.2 Example: Energetic particles in the heliosphere

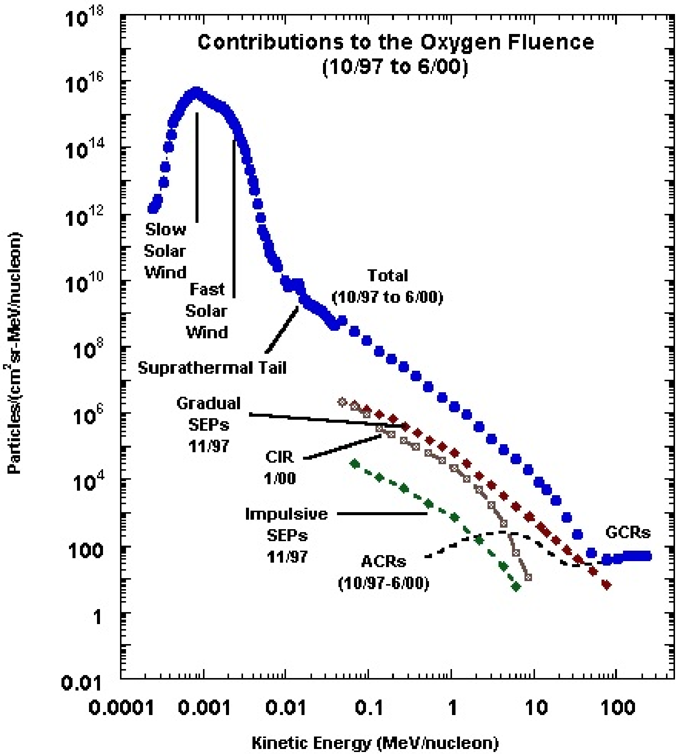


Figure 10.2: *Oxygen energy distribution. Energetic particles in the heliosphere originate from a number of separate sources and acceleration processes. From NASA Advanced Composition Explorer (ACE) s/c*

10.3 Kinetic equation

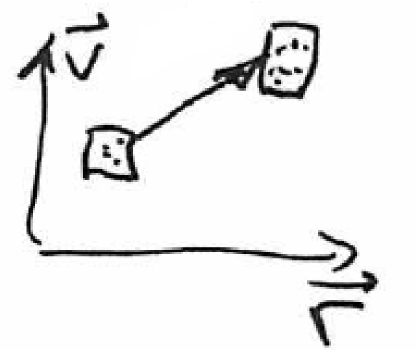


Figure 10.3: *The density in phase space is conserved as the system evolves without collisions.*

If we ignore collisions and particle sources/sinks and consider a closed system, the distribution function obeys **Liouville theorem**, as a result of which we can write

$$\text{phase-space density conservation} \iff \frac{df}{dt} = 0 \tag{10.2}$$

where the time derivative is along a trajectory in the **phase space** $(\vec{\mathbf{v}}, \vec{\mathbf{r}})$. Taking the derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{\mathbf{r}}}{dt} \cdot \frac{\partial f}{\partial \vec{\mathbf{r}}} + \frac{d\vec{\mathbf{v}}}{dt} \cdot \frac{\partial f}{\partial \vec{\mathbf{v}}} = 0$$

and derive kinetic equation

$$\frac{\partial f}{\partial t} + \vec{\mathbf{v}} \cdot \frac{\partial f}{\partial \vec{\mathbf{r}}} + \frac{\vec{\mathbf{F}}}{m} \cdot \frac{\partial f}{\partial \vec{\mathbf{v}}} = 0 \tag{10.3}$$

If the force $\vec{\mathbf{F}}$ is entirely electromagnetic, the equation takes the form

$$\frac{\partial f}{\partial t} + \vec{\mathbf{v}} \cdot \frac{\partial f}{\partial \vec{\mathbf{r}}} + \frac{q}{m}\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot \frac{\partial f}{\partial \vec{\mathbf{v}}} = 0 \tag{10.4}$$

when the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are the average of electric and magnetic fields from all the particles in plasma, the equation is called **Vlasov** equation.

The kinetic equation (10.4) should be completed with the system of Maxwell equations to find $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ from f.

10.4 Collisions in kinetic equation

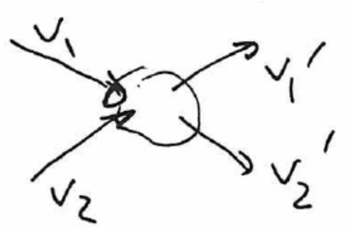


Figure 10.4: *The changes in v as disappearance of old and appearance of new particles, hence uncorrelated.*

When there are **collisions** in plasma, $df/dt \neq 0$, and we include collisions using collision integral - the rate of change of f due to collisions

$$\frac{\partial f}{\partial t} + \vec{\mathbf{v}} \cdot \frac{\partial f}{\partial \vec{\mathbf{r}}} + \frac{\vec{\mathbf{F}}}{m} \cdot \frac{\partial f}{\partial \vec{\mathbf{v}}} = St(f) = C(f) = \left(\frac{\partial f}{\partial t}\right)_C \tag{10.5}$$

This equation is known as **Boltzmann equation**, St is short from 'stoss', and C for collision.

The kinetic equation can be modified to include sources or sinks of particles aading terms at the RHS of equation (10.5).

When there are collisions we need to take into account changes in particle density in phase space.

It could be insightful to look at a simplified form of a collisional integral. Sometimes the collision term can be approximated via simple form:

$$\left(\frac{\partial f}{\partial t}\right)_C = \frac{f_{eq} - f}{\tau} \tag{10.6}$$

where f_{eq} is equilibrium distribution (often Maxwellian), τ is the collision time. This is called a **Krook collision term**.

Then the integral shows that the distribution function f relaxes towards equilibrium distribution f_{eq} over time τ. Indeed, a small perturbation of the distribution function $f_1 = f - f_{eq}$ evolves

$$\frac{df_1}{dt} = \frac{f_{eq} - f}{\tau} = -\frac{f_1}{\tau}$$

and disappears over time τ.

10.5 Plasma waves in collisionless plasma

As an illustration of the use of the Vlasov equation, we consider electron plasma oscillations in a uniform plasma without external fields $\vec{\mathbf{B}} = 0$ and $\vec{\mathbf{E}} = 0$. Let us consider, in the first order, the perturbation in f:

$$f(\vec{\mathbf{r}}, \vec{\mathbf{v}}, t) = f_0(\vec{\mathbf{r}}, \vec{\mathbf{v}}, t) + f_1(\vec{\mathbf{r}}, \vec{\mathbf{v}}, t) + ...$$

The first order Vlasov equation (10.4)

$$\frac{\partial f_1}{\partial t} + \vec{\mathbf{v}} \cdot \frac{\partial f_1}{\partial \vec{\mathbf{r}}} + \frac{q}{m}\vec{\mathbf{E}}_1 \cdot \frac{\partial f_0}{\partial \vec{\mathbf{v}}} = 0 \tag{10.7}$$

where $\vec{\mathbf{E}}_1$ is the perturbation electric field. Poisson equation $\epsilon_0 \nabla \cdot \vec{\mathbf{E}} = qn$:

$$\epsilon_0 \nabla \cdot \vec{\mathbf{E}}_1 = -en_1 = -e\int f_1 d^3 v$$

We assume the ions are massive and fixed and the waves are plasma waves in x-direction

$$f_1 \to f_1 \exp(-i\omega t + ikx)$$
$$E_1 \to E_1 \exp(-i\omega t + ikx)$$

then we can write:

$$-i\omega f_1 + ikv_x f_1 = \frac{e}{m} E_1 \frac{\partial f_0}{\partial v_x} \tag{10.8}$$

$$\epsilon_0 ikE_1 = -e \int f_1 d^3 v \tag{10.9}$$

combining (10.8,10.9) we obtain

$$1 = -\frac{e^2}{km\epsilon_0} \int \frac{\partial f_0}{\partial v_x} \frac{d^3 v}{\omega - kv_x} = \tag{10.10}$$

$$= -\frac{e^2}{km\epsilon_0} \int_{-\infty}^{+\infty} dv_z \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v_x} \frac{dv_x}{\omega - kv_x}. \tag{10.11}$$

If f_0 is a maxwellian, the integration over v_z, v_y can be carried out

$$f_0(v_x) = n_0 \frac{1}{(2\pi k_B T/m)^{1/2}} \exp\left(-\frac{m v_x^2}{2k_b T}\right) \qquad (10.12)$$

taking normalised function $\tilde{f}_0 = f_0/n_0$, we find

$$1 = \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0}{\partial v_x} \frac{dv_x}{v_x - \omega/k}$$

which is the equation for the **dispersion relation, $\omega(k)$**.

11 Electron plasma waves and Landau damping

LECTURE OUTLINE

- Electron plasma waves and Landau damping
- Dispersion relation for Langmuir waves
- Physics of Landau damping

11.1 Dispersion relation for electron plasma waves

Let us consider the dispersion relation from the last lecture:

$$1 = \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0}{\partial v_x} \frac{dv_x}{v_x - \omega/k} \tag{11.1}$$

To find $\omega(k)$, we need to evaluate the integral.

However, the integral in equation (11.1) is not easy to evaluate because of the singularity $v_x = \omega/k$.

11.2 Landau rule

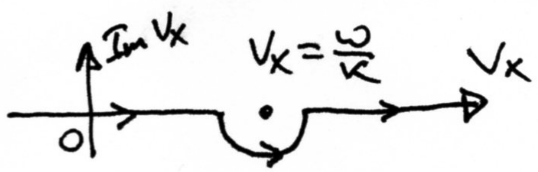


Figure 11.1: *The integration is* ***below*** *the pole at* $v_x = \omega/k$

Let us split the integral in equation (11.1) into three parts. Landau (1946) suggested the rule for avoiding the poles (adding small positive imaginary part):

$$\omega \rightarrow \omega + i0$$

and will integrate along the path shown Figure 11.1, where v_x is a real part of complex variable z. As shown in Figure (11.1), the integral can be written as the sum:

$$\int\limits_{-\infty}^{+\infty} \frac{\frac{\partial \tilde{f}_0}{\partial v_x} dv_x}{v_x - \omega/k} = \int\limits_{-\infty}^{v_x-\omega/k-\rho} \frac{\frac{\partial \tilde{f}_0}{\partial v_x} dv_x}{v_x - \omega/k} + \int\limits_{v_x-\omega/k+\rho}^{+\infty} \frac{\frac{\partial \tilde{f}_0}{\partial v_x} dv_x}{v_x - \omega/k} + \int\limits_{C} \frac{\frac{\partial \tilde{f}_0}{\partial v_x} dz}{z - \omega/k} \tag{11.2}$$

The third integral is taken along the **semi-circle** of radius $\rho \to 0$ in the complex plane (Figure 11.1). Using complex number substitute $z = \omega/k + \rho e^{i\phi}$

$$\lim_{\rho\to 0}\int_C \frac{\partial \tilde{f}_0}{\partial v_x}\frac{dz}{z-\omega/k} = \lim_{\rho\to 0}\int_{\pi}^{2\pi} \frac{\partial \tilde{f}_0}{\partial v_x}\frac{i\rho e^{i\phi}d\phi}{\rho e^{i\phi}} = i\pi \left.\frac{\partial \tilde{f}_0}{\partial v_x}\right|_{v_x=\omega/k}$$

In the limit $\rho \to 0$, the first two integrals in Equation (11.2) give the **Cauchy principal value** of the integral. Hence we have :

$$\int_{-\infty}^{+\infty} \frac{\frac{\partial \tilde{f}_0}{\partial v_x}dv_x}{v_x-\omega/k} = \mathrm{PV}\int_{-\infty}^{+\infty} \frac{\frac{\partial \tilde{f}_0}{\partial v_x}dv_x}{v_x-\omega/k} + i\pi \left.\frac{\partial \tilde{f}_0}{\partial v_x}\right|_{v_x=\omega/k}, \qquad (11.3)$$

where PV denotes the principal value of the integral.

Using the integral with Landau rule applied (equation 11.3), the dispersion relation can written:

$$1 = \frac{\omega_{pe}^2}{k^2}\left(\mathrm{PV}\int_{-\infty}^{+\infty}\frac{\partial \tilde{f}_0}{\partial v_x}\frac{dv_x}{v_x-\omega/k} + i\pi\frac{\partial \tilde{f}_0}{\partial v_x}\bigg|_{v_x=\omega/k}\right) \tag{11.4}$$

where we have both real (first term) and imaginary (second term) parts.

11.3 Real part of the dispersion relation

The integral in the real part of Equation (11.4) can be evaluated by integration by parts

$$\mathrm{PV}\int_{-\infty}^{+\infty}\frac{\partial \tilde{f}_0}{\partial v_x}\frac{dv_x}{v_x-\omega/k_x}=\left.\frac{\tilde{f}_0}{v_x-\omega/k}\right|_{-\infty}^{+\infty}+\mathrm{PV}\int_{-\infty}^{+\infty}\frac{\tilde{f}_0 dv_x}{(v_x-\omega/k)^2}, \quad (11.5)$$

where the first term on the right hand side is zero and we only need to evaluate the second integral.

We can assume large phase velocities, i.e. $\omega/k \gg v_x$, then we can expand $(v_x-\omega/k)^{-2}$:

$$(v_x-\omega/k)^{-2}=\left(\frac{\omega}{k}\right)^{-2}\left(1-\frac{kv_x}{\omega}\right)^{-2}=\left(\frac{\omega}{k}\right)^{-2}\left(1+2\frac{kv_x}{\omega}+3\frac{k^2v_x^2}{\omega^2}+\ldots\right)$$

Taking real integral (11.5) with the expansion

$$\mathrm{PV}\int_{-\infty}^{+\infty}\frac{\tilde{f}_0 dv_x}{(v_x-\omega/k)^2}=\left(\frac{\omega}{k}\right)^{-2}\mathrm{PV}\int_{-\infty}^{+\infty}\tilde{f}_0\left(1+2\frac{kv_x}{\omega}+3\frac{k^2v_x^2}{\omega^2}+\ldots\right)dv_x$$

The odd terms in v_x will vanish and $\int \tilde{f}_0 v_x^2 dv_x$ is just the variance.

Since $\tilde{f}_0$ is Maxwellian (from Equation 10.12) $\int \tilde{f}_0 v_x^2 dv_x = k_B T/m$. Then we can write:

$$1 = \frac{\omega_{pe}^2}{k^2}\left[\left(\frac{\omega}{k}\right)^{-2}\left(1 + 3\frac{k^2 k_B T}{m\omega^2}\right)\right] = \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2 k_B T}{m\omega^2}\right)$$

if the thermal correction is small, we can replace ω^2 by ω_{pe}^2 in the second term.

Hence we find

$$\omega^2(k) = \omega_{pe}^2 + 3\frac{k^2 k_B T}{m} \tag{11.6}$$

which is the **dispersion relation for Langmuir waves.** These waves are sometimes called electron plasma oscillations.

11.4 Phase and group speeds of Langmuir waves

The frequency of these waves is near ω_{pe}, the **phase speed** is $v_{ph} = \omega/k \simeq \omega_{pe}/k$. The **group speed** can be calculated approximating $\omega(k) \simeq \omega_{pe}(1 + 3\frac{k^2 k_B T}{2m\omega_{pe}^2})$:

$$\frac{\partial \omega}{\partial k} \simeq 3\frac{k k_B T}{m\omega_{pe}} \simeq 3\frac{k_B T}{m}\frac{1}{v_{ph}} \simeq 3v_{Te}\frac{v_{Te}}{v_{ph}}$$

which is of the order of (or smaller than) v_{Te} but $\neq 0$. Because of the relatively small group velocity, Langmuir waves (electron plasma waves) are called 'oscillations' or electron plasma oscillations.

Langmuir waves are frequently observed in space plasmas, e.g. solar wind (Figure 11.2), ionosphere, magnetosphere of the Earth and other planets.

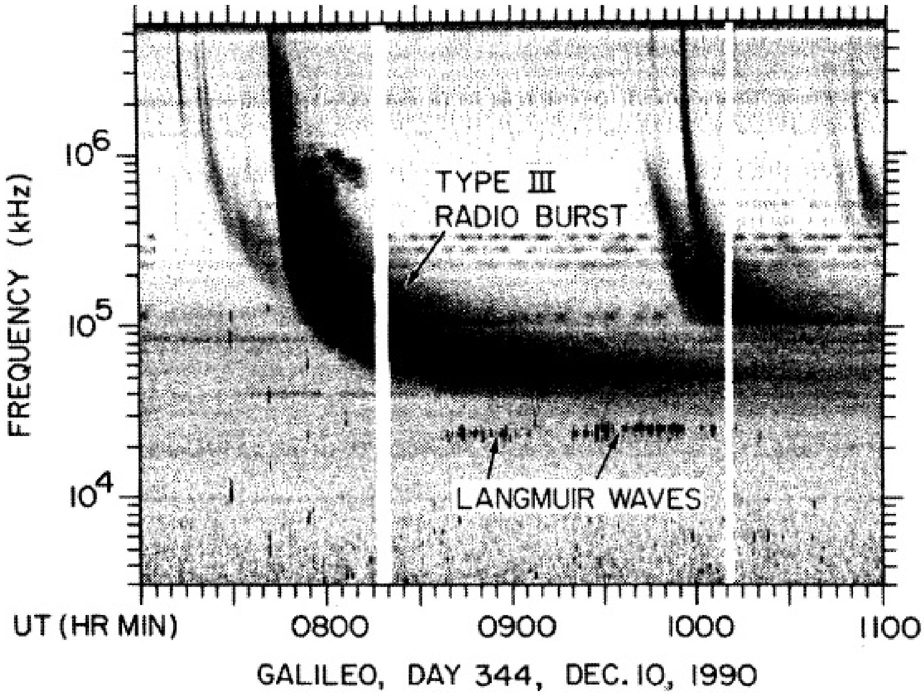


Figure 11.2: *The bursty narrow band emission near 23 KHz (solar wind electron plasma frequency $f_{pe} \simeq$ 23 kHz) are the Langmuir waves responsible for generation of type III solar radio emission. From Gurnett et al, 1993*

11.5 Landau damping

Let us now consider the **imaginary part** in Equation (11.4). For simplicity, let us ignore the thermal correction in the real part, so Langmuir wave dispersion relation $\omega(k) \simeq \omega_{pe}$, then we have

$$1 = \frac{\omega_{pe}^2}{\omega^2} + i\pi \frac{\omega_{pe}^2}{k^2} \frac{\partial \tilde{f}_0}{\partial v_x}\bigg|_{v_x=\omega/k}$$

hence

$$\omega^2 = \omega_{pe}^2 \left(1 - i\pi \frac{\omega_{pe}^2}{k^2} \frac{\partial \tilde{f}_0}{\partial v_x}\bigg|_{v_x=\omega/k} \right)^{-1}$$

Assuming the imaginary part is small, we find the dispersion relation with the imaginary term

$$\omega(k) \simeq \omega_{pe} + i\gamma_k = \omega_{pe} + i\omega_{pe}\frac{\pi}{2}\frac{\omega_{pe}^2}{k^2}\frac{\partial \tilde{f}_0}{\partial v_x}\bigg|_{v_x=\omega/k} \qquad (11.7)$$

where the imaginary part γ_k is the **Landau damping rate**.

11.6 Landau damping in Maxwellian plasma

Let us consider Maxwellian plasma with distribution from Equation (10.12) (considering 1D perturbation and $v_x = v$ hereafter)

$$\tilde{f}_0(v_x) = (\pi v_{Te}^2)^{-1/2}\,\mathrm{exp}\left(-\frac{v^2}{v_{Te}^2}\right),$$

where $v_{Te} = \sqrt{2k_B T/m}$. Then the derivative $\partial \tilde{f}_0(v_x)/\partial v_x$ follows

$$\frac{\partial \tilde{f}_0}{\partial v_x} = -\frac{2v}{v_{Te}^2}(\pi v_{Te}^2)^{-1/2}\,\mathrm{exp}\left(-\frac{v^2}{v_{Te}^2}\right) = -\frac{2v}{\sqrt{\pi}v_{Te}^3}\,\mathrm{exp}\left(-\frac{v^2}{v_{Te}^2}\right)$$

Using $v = \omega/k$, the damping rate from 11.7 is (note that the thermal correction from Langmuir wave dispersion 11.6 is retained in the exponent only)

$$\gamma_k \simeq \sqrt{\pi}\omega_{pe}\left(\frac{\omega_{pe}}{kv_{Te}}\right)^3 \exp\left(-\frac{\omega^2(k)}{k^2 v_{Te}^2}\right) \simeq \tag{11.8}$$

$$\sqrt{\pi}\omega_{pe}\left(\frac{\omega_{pe}}{kv_{Te}}\right)^3 \exp\left(-\frac{\omega_{pe}^2}{k^2 v_{Te}^2}\right)\exp\left(-\frac{3}{2}\right) \simeq \tag{11.9}$$

$$\simeq 0.22\sqrt{\pi}\omega_{pe}(k\lambda_{De})^{-3} \exp\left(-\frac{1}{2k^2\lambda_{De}^2}\right) \tag{11.10}$$

where we used that $\omega_{pe}^2 = e^2 n/(\epsilon_0 m_e)$ and $\lambda_{De}^2 = \epsilon_0 k_B T/ne^2$, $v_{Te}^2/2 = k_b T/m_e = \lambda_{De}^2\omega_{pe}^2$.

Hence Langmuir waves should be strongly absorbed in Maxwellian plasma (**without collisions**) for $kv_{Te} \gtrsim \omega_{pe}$. Langmuir Waves with $kv_{Te} \ll \omega_{pe}$ are weakly damped.

11.7 Resonance condition

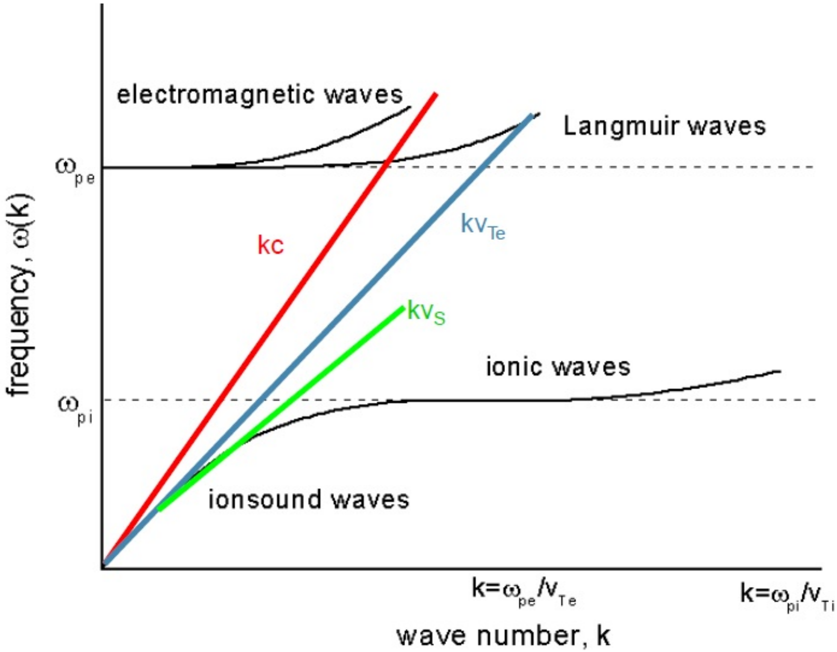
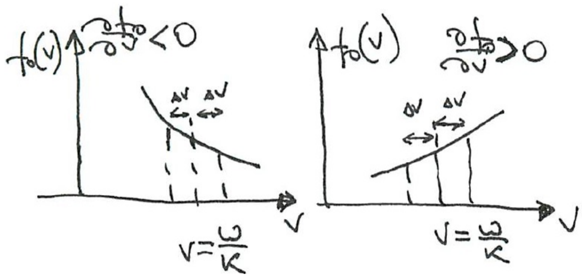


Figure 11.3: *The resonant condition is when the wave has zero frequency in the rest frame of particle $\omega(k) = kv_x$. This resonance is called* ***Cherenkov resonance****.*

11.8 Physics of Landau damping



The expression for Landau damping (11.7) says that there is no wave energy loss/gain when $\partial f/\partial v = 0$. Electrons with $v + \Delta v$ give energy to the wave $v = \omega/k$ and the electrons $v-\Delta v$ take anergy from the wave . If there is larger number of electrons at $v-\Delta v$, i.e. $\partial f/\partial v < 0$, the larger number of the electrons will be accelerated then decelerated and hence the **wave energy is absorbed by the particles.**

When $\partial f/\partial v > 0$, the larger number of electron is at $v+\Delta v$, so the electrons will, on average, lose the energy to the wave and there is a transfer of energy from the particles to the electric field or **wave growth**. Note that there is no phase in the expression for the resonance $v = \omega/k$.

11.9 Landau damping and generation of waves

The damping is not randomisation by collisions, but a resonant $v = v_{ph} = \omega/k$ transfer of energy from waves to particles in collisionless plasma. It can be reversed if $\partial \tilde{f}_0/\partial v_x > 0$.

Such resonant interaction (i.e. the phase velocity of waves is the same as velocity of interacting particles) appears for other plasma waves, not only Langmuir waves considered. Ions can also resonantly interact with plasma waves via such mechanism. The Landau damping is an example of **wave-particle interactions**, which play the key role in collisionless plasma.

11.10 Example: electrons and Langmuir waves in the solar wind

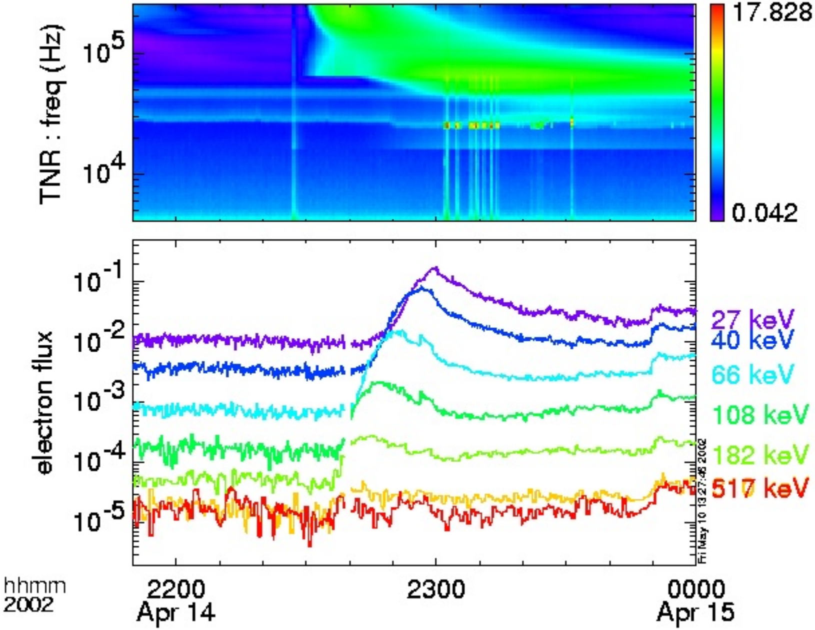


Figure 11.4: *Langmuir waves in solar wind (solar wind electron plasma frequency $f_{pe} \simeq 20$ kHz) associated with solar energetic electrons near the Earth. Figure from Krucker et al, 2007*

Langmuir waves play an important role in solar radio emission.