University of Glasgow

Department of Physics and Astronomy

Problems Handbook 2005–2006

Notation: Marks are shown in square brackets in the right-hand margin. Question numbers followed by a small 'e' are recent exam questions. Answers to some numerical problems are shown in curly brackets.

You may need to use some of the following constants in your solutions to the problems:

Values of constants

speed of light	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
gravitational constant	G	$6.673 \times 10^{-11}\,\mathrm{N\,m^{2}\,kg^{-2}}$
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
Boltzmann constant	k	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	$5.671 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Rydberg constant	R_{∞}	$1.097 \times 10^{7}\,\mathrm{m^{-1}}$
Avogadro constant	N_{A}	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
gas constant	R	$8.315\,\mathrm{J\,mol^{-1}\,K^{-1}}$
proton mass	m_{p}	$1.673 \times 10^{-27}\,\mathrm{kg}$
electron mass	m_{e}	$9.109 \times 10^{-31}\,\mathrm{kg}$
elementary charge	e	$1.602 \times 10^{-19}\,\mathrm{C}$
electronvolt	eV	$1.602 \times 10^{-19}\,\mathrm{J}$
astronomical unit	AU	$1.496 \times 10^{11}\,\mathrm{m}$
parsec	pc	$3.086 \times 10^{16}\,\mathrm{m}$
light year	ly	$9.461 \times 10^{15}\,\mathrm{m}$
solar mass	$M_{\odot}$	$1.989 \times 10^{30}\,\mathrm{kg}$
solar radius	$R_{\odot}$	$6.960 \times 10^{8}\,\mathrm{m}$
solar luminosity	$L_{\odot}$	$3.826 \times 10^{26}\,\mathrm{W}$
Earth mass	$M_{\oplus}$	$5.976 \times 10^{24}\,\mathrm{kg}$
Earth radius	$R_{\oplus}$	$6.378 \times 10^{6}\,\mathrm{m}$
obliquity of the ecliptic	ϵ	$23^{\circ}\,26'$

Contents

1 General Astronomy

A1X – General Astronomy

1.1e Describe a method of determining the radius of a planetary orbit (or semi-major axis)

(a) as a multiple of the Earth's orbital radius

(b) absolutely – i.e., in metres. [5]

1.2e Explain with the aid of orbital diagrams how the loop-like motions of an outer planet as seen against the stellar background can be explained using

(a) the Ptolemaic system

(b) the heliocentric system. [5]

1.3 Define the term *Astronomical Unit* (AU). [3]

The maximum angular elongation of Venus from the Sun, seen from Earth, is $\eta_{\max} = 46°$. Find the radius of Venus's orbit (assumed circular) in AU. {0.719 AU} [4]

If at this configuration a radar signal takes $11^{\mathrm{m}}35^{\mathrm{s}}$ for the Earth-Venus round trip, find the value of 1AU in metres. (Neglect the motion of Venus and the Earth during the time of the radar signal round-trip.) {1.499×10^{11} m} [5]

Explain, with a diagram, why successive conditions of maximum elongation do not provide the same measured values of $\eta_{\max}$. [3]

Apply Kepler's 3rd Law of Planetary Orbits ($a^3 \propto P^2$) to find the shortest radio communication time from the Earth to the Voyager probe when at Neptune (orbital period 164.3 yr). {$4^{\mathrm{h}}1^{\mathrm{m}}30^{\mathrm{s}}$} [5]

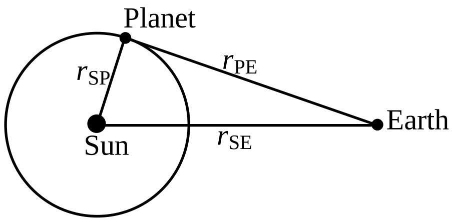
1.4 The orbital periods of Venus, Jupiter and Saturn around the Sun are given in the table below. Calculate their mean distances (in AU) from the Sun.

	Orbital period	Minimum angular diameter	Distance from Sun	Maximum angular diameter	Physical diameter	Maximum distance from Earth
Venus	225 days	10″				
Jupiter	12 years	30″				
Saturn	30 years	15″				

Explain why the angular diameters of the planets vary with their phase. The minimum angular size of Venus is 10″. At what phase does this take place? What is the appropriate maximum angular size of Venus? Calculate the physical diameter of Venus.

The table gives the approximate minimum angular sizes of Jupiter and Saturn. What are their respective maximum angular sizes? Calculate their physical diameters.

Estimate the flux of the reflected light to arrive at the Earth from each of the planets when it is furthest away from the Earth as a fraction of the direct flux of the Sun. (Consider both inferior and superior planets.)



The differences in apparent magnitude of two celestial bodies is given in terms of their respective fluxes by

$$m_1 - m_2 = -2.5 \log \frac{F_1}{F_2}$$

Estimate the difference in apparent magnitude of Jupiter at maximum and minimum brightness.

Discuss:

- When do you expect Venus to be brightest?
- Other stars may have planetary systems. Would you expect the occultation of the star by a planet to produce a change in the star's apparent magnitude?
- Write down an expression for the change in magnitude, and sketch how this would vary with time. Would such a change be measurable?
- How might you distinguish the variation in apparent magnitude due to occultation from the variation produced by possible star spots?

1.5e On occasion, Eros, a minor planet, approaches the Earth to within a distance of 30×10^6 km. It is suggested that its distance might be measured absolutely by parallax using two observatories separated by 1 000 km. If the accuracy of positional measurement is ± 0.001 arcsec, estimate the uncertainty in the distance determination. [5]

[1 radian = 206 265 arcsec]

A1Y – General Astronomy

1.6e Two stars radiate as black bodies with the maximum brightness at wavelengths, λ_{max}, of 500 nm and 700 nm. It is also known that the first star has a radius three times larger than the second. What is ratio of their luminosities? [5]

1.7 A visual double star has a measured parallax of 0.105 arcsec and the semi-major axis of the orbit subtends 21.47 arcsec. If the orbital period is 350 years, calculate the sum of the masses of the stars. {$69.79 M_\odot$} [8]

From measurements of the system's proper motion, it was found that at one epoch the subtended angles from the centre of gravity were 3.84 and 7.12 arcsec. Determine the individual masses of the stars. {$45.3 M_\odot$; $24.49 M_\odot$} [12]

1.8 (a) A star of apparent magnitude 6 shows an annual parallax of $0.02''$. What is its distance in parsecs, AU and metres? {50 pc; 1.03×10^7 AU; 1.55×10^{18} m}

(b) A second star, with an identical spectrum, has apparent magnitude 11. Find its distance in parsecs. { 500 pc}

(c) Find the absolute magnitude of each star. {2.5}

1.9 The 'hydrogen-alpha' spectrum line of hydrogen ($\lambda_0 = 656.3$ nm) is observed from two galaxies. In the first it is seen at $\lambda = 694.2$ nm and in the second at $\lambda = 765.1$ nm. Find the ratio of the distances of these two galaxies, using Hubble's law. Taking values of H and of c from the front cover, find the distances in Mpc, and as fractions of the (Hubble) radius of the visible Universe. {2.87; 0.0577; 0.166}

1.10 (a) Two stars provide a flux ratio of 100:1. What is their apparent difference of magnitude? {5}

(b) A solar-type star has an absolute magnitude of 4.8. It appears in a catalogue as having an apparent magnitude of 9.6. What is its distance? {91.3 parsec}

1.11 An X-ray source has a luminosity at 2 keV of $10^{26} W$. Convert 2 keV into Joules and hence calculate the photon production rate (number of photons per second) emitted by this source at this energy.

1.12 A radio telescope is tuned to operate at 3 000 MHz. Calculate

(a) the wavelength of the received radiation

(b) the energy in eV of the photons at this frequency.

1.13 A star is measured to have a parallax of 0.045 arcsec. Calculate its distance in parsecs, metres and light years.

2 Positional Astronomy

2.1e Draw a celestial sphere for north latitude 45°, indicating clearly the zenith point, the north celestial pole and the horizon. Mark in the cardinal points of the horizon and draw in the celestial equator. The date is March 21; mark in the Sun's position at noon. [5]

2.2e List Galileo's principal astronomical observational discoveries. Explain which of these could be used as evidence to confirm the heliocentric theory of Copernicus. [5]

2.3e Draw a celestial sphere to indicate clearly the relationship between the equatorial coordinate system of right ascension and declination and the coordinate system of ecliptic latitude and longitude. Indicate the Sun's position on 21 June. What is its declination at this time? [5]

2.4e Draw a celestial sphere for an observer on the Earth's equator. Mark in the north and south celestial poles and the celestial equator. Mark in the Sun's approximate position at noon on 21st December. [5]

2.5e Indicate clearly on a diagram the position of an inferior planet with respect to the Earth and the Sun for the following configurations:

(a) Inferior conjunction

(b) Superior conjunction

(c) Maximum elongation east

(d) Maximum elongation west. [5]

2.6e Define a great circle on the surface of a sphere. Explain why this type of curve is particularly important. What is the length of the great circle arc joining the two points on the surface of the Earth with geographical latitude and longitude (60° N, 90° E) and (60° N, 90° W)? [5]

2.7 Two places A and B on the same parallel of latitude 38°33′ N are 123°19′ apart in longitude. Calculate, in nautical miles,

(a) their distance apart along the parallel

(b) the great circle distance AB.

{5786.5 nmi; 5219.7 nmi}

2.8e Define carefully the coordinate system of terrestrial latitude and longitude and compare it with the celestial coordinate system of right ascension and declination. [6]

Derive a formula to give the shortest distance between two points on the same parallel of latitude ϕ which differ in longitude by an amount $\Delta\lambda$. Explain carefully how this distance may be expressed in nautical miles. [5]

A star is observed at the zenith at Glasgow University Observatory (55°54′ N, 4°18′ W). At the same instant of time a second star is at the zenith at the University of Helsinki Observatory (60°09′ N, 24°57′ E). Calculate the right ascension and the declination of each star if the Greenwich sidereal time at this instant is 16^h41^m. [6]

2.9e Define carefully the terms conjunction, opposition and quadrature as they are applied to a superior planet. Define further the sidereal period and the synodic period for such a planet and explain how they are related. [5]

The planet Mars moves round the Sun in an orbit of semi-major axis 1.52 astronomical units. Calculate its sidereal period of revolution in years. [2]

Making the approximation that the orbits of Mars and the Earth are circular and coplanar, calculate the synodic period of Mars in years and the interval of time in days between opposition and quadrature for this planet. [7]

Mars is observed at quadrature on March 21. Calculate its right ascension and declination at this time. [3]

(Obliquity of the ecliptic $\epsilon = 23°26'$)

2.10e Define carefully what is meant by a *parallel of latitude* and a *meridian of longitude*, illustrating your definition with a diagram. Explain the conventions that are used in assigning a latitude and a longitude to a position on the surface of the Earth. Give a formal definition of the *nautical mile*. [6]

State, without proof, the *cosine formula* of spherical trigonometry. [2]

Prove that the shortest distance between the two points P_1 and P_2 on the Earth's surface with geographical coordinates (ϕ_1, λ_1) and (ϕ_2, λ_2), latitude and longitude respectively, is given by

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2).$$ [4]

Find the length in nautical miles of the shortest air route from Dunedin (45°51′ S, 170°30′ E) to Los Angeles (33°51′ N, 118°21′ W). [5]

2.11e Define carefully the coordinate system of *ecliptic longitude* and *ecliptic latitude*. Indicate how a star's ecliptic coordinates (λ, β) are related to its right ascension and declination (α, δ) by identifying the appropriate spherical triangle on the celestial sphere. In particular prove that

$$\cos\lambda \cos\beta = \cos\alpha \cos\delta.$$ [8]

Estimate the date when the Sun's ecliptic longitude is 45° and calculate the Sun's right ascension and declination for that date. [12]

(Obliquity of the ecliptic, $\epsilon = 23.5°$)

2.12 An aircraft leaves Lima (12°10′ S, 77°05′ W) and flies directly to Rome (41°53′ N, 12°33′ E). Draw a diagram marking the position of Lima and Rome, and the great circle joining them.

(a) Calculate the distance travelled in nautical miles.

(b) Calculate the aircraft's bearing as it approaches Rome.

(c) Determine the longitude of the point on the flight where the aircraft crosses the Equator.

{5869 nmi, 99°.357 69, 63°35′}

2.13 Prove that, in an equilateral spherical triangle, the sides and angles satisfy the condition that $\sec A - \sec a = 1$. [20]

2.14 Two seaports are on the same parallel of latitude 42°27′ N. Their difference in longitude is 137°36′. Ship *A* sails along the parallel of latitude from one port to the other, while ship *B* follows the most direct great circle route. Ship *B* sails at a constant speed of 20 knots. How long will it take to complete the voyage? {$10^{d}20^{h}48^{m}$} [12]

What average speed will ship *A* have to maintain to complete its voyage in the same time as ship *B*? {23.36 knots} [8]

2.15 Calculate the length in degrees of the great circle arc joining San Francisco (37°40′ N, 122°25′ W) and Tokyo (35°48′ N, 139°45′ E). {76.95°} [9]

What is the shortest distance between these two cities in nautical miles? {4617.03 nmi} [2]

An aircraft leaves San Francisco at 10 pm local time on August 21 and flies directly to Tokyo maintaining an average speed of 480 knots. Calculate

(a) the duration of the flight {9^h18^m} [3]

(b) the local time and date of arrival in Tokyo. {00:18} [6]

2.16 Using the result of the previous question, determine the direction in which an aircraft should depart from San Francisco to follow the most direct route to Tokyo. Calculate further the latitude and longitude of the most northerly point that will be reached on this direct route. {$56°34'$ W of N; $48°39'$ N; $169°38'$ W}

2.17 The altitudes of a star at upper and lower transits (both north of the zenith) are $65°23'$ and $14°01'$. Find the latitude of the observer and the star's declination and calculate the star's altitude, azimuth and hour angle when it is at its maximum azimuth west. {$\delta = 64°19'$; $\phi = 39°42'$; $a = 45°08'$; $A = 37°17'$}

2.18 Calculate the azimuth of the Sun at rising on Midsummer's day at Stonehenge (latitude $50°10'$ N) at a time when the obliquity of the ecliptic was $23°48'$.

2.19 Describe the celestial sphere and the diurnal motions of the stars as they would appear for an observer at

(a) the Earth's North Pole

(b) a point on the Earth's equator.

2.20 Prove that the maximum azimuth (east or west of north) of a circumpolar star is

$$A = \sin^{-1}(\cos\delta \sec\phi).$$

The right-hand side of this equation is undefined when $\delta < \phi$. How do you account for this?

2.21 Define the *phase angle*, ϕ, and the *elongation*, η, of a planet illustrating your definition with diagrams for an inferior and a superior planet. [5]

Why is the phase angle so called? [3]

Assuming that the Moon is at a distance of 3.84×10^5 km from the Earth, calculate its elongation to the nearest arcminute when it is observed to be exactly half illuminated.$89°51'$ [12]

2.22 An asteroid is moving in an orbit of semi-major axis 2.87 astronomical units. Calculate its sidereal period of revolution. {4.86 yr} [3]

Assuming that the asteroid is moving in a circular orbit in the ecliptic plane, calculate the synodic period in years. {1.26 yr} [3]

What is the maximum phase angle and the minimum phase for this asteroid? {$20.39°$; 0.969} [5]

Work out the interval of time in days between opposition and the occurrence of this minimum phase. {0.24 yr} [9]

2.23 Draw a celestial sphere for an observer at latitude 56° N indicating the *horizon* and the *celestial equator*. Mark in the north, south and west points of the horizon and also the *north celestial pole* and the *zenith point*. [6]

A star has declination 18°. Indicate its approximate position on the diagram when it is setting and calculate its azimuth at that instant. {303°33′} [7]

What is the hour angle of the star when it sets? {7^h55^m} [5]

How long is the star above the horizon in

(a) sidereal time? {15^h50^m} [1]

(b) solar time? {15^h48^m} [1]

2.24 (a) What is the ecliptic latitude and longitude of the Sun on May 1st?

(b) Calculate the Sun's right ascension and declination for this date, assuming that the obliquity of the ecliptic $\epsilon = 23°.5$.

(c) Calculate the hour angle of the Sun at sunset for this date for an observer at Glasgow, latitude 56°N, longitude 4°15′W.

(d) What is the interval (in solar time) between sunrise and sunset for this observer on this date? Determine the Greenwich mean times of sunset and sunrise.

(e) What is the local sidereal time and what is the Greenwich sidereal time of sunset for the Glasgow observer on May 1st?

2.25 The planet Jupiter is moving in an orbit of semimajor axis 5.202 astronomical units. Calculate its *sidereal* period in years.

Calculate the *synodic* period of Jupiter in days making the approximation that both Jupiter and the Earth move round the sun in circular coplanar orbits.

Calculate the interval of time in days during which Jupiter's sidereal motion will be *retrograde*.

2.26$_e$ Draw a celestial sphere for an observer on the Earth's equator. Indicate the positions of the north and south celestial poles and draw in the Sun's diurnal path for June 21 and December 21. [5]

2.27$_e$ Show that the sidereal motion of a superior planet must be retrograde at opposition and direct at quadrature. [5]

2.28$_e$ Draw a geocentric celestial sphere, marking clearly the celestial equator and ecliptic. State the Sun's right ascension, declination, ecliptic longitude and ecliptic latitude for each of the following dates: [1]

(a) March 21 [1]

(b) June 21 [1]

(c) September 21 [1]

(d) December 21. [1]

2.29[e] Explain carefully how the *latitude* and *longitude* of a point on the Earth's surface are defined. [4]

Two points on the Earth's surface have latitude and longitude (ϕ_1, λ_1) and (ϕ_2, λ_2) respectively. Prove that the shortest distance between them is d, where

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2).$$

How would you express this distance in nautical miles? [5]

On March 21 the planet Mercury had right ascension and declination $(0^h 45^m, 5°04')$. Calculate its elongation from the Sun at that time. [8]

2.30[e] Define precisely what is meant by a *circumpolar* star. Show that the condition for a star to be circumpolar is that its declination δ satisfies the inequality

$$\delta > 90° - \phi$$

where ϕ is the latitude of the observing site. Show further that if $\phi > 45°$ than all stars that transit north of the zenith are circumpolar. [5]

Show that if the Sun becomes circumpolar for part of the year for all points within the arctic circle, i.e. $\phi > (90° - \epsilon)$, where ϵ is the obliquity of the ecliptic. Show further that this period of continuous daylight (midnight Sun) persists while the ecliptic longitude λ of the Sun satisfies the equation

$$\sin\lambda > \cos\phi \operatorname{cosec}\epsilon \qquad [6]$$

Calculate the length of the period of continuous daylight for an observer at latitude 72°, on the assumption that the ecliptic longitude of the Sun increases strictly uniformly during the year. [6]

[Obliquity of the ecliptic $\epsilon = 23°26'$.]

2.31 Define what is meant by a circumpolar star using a diagram to illustrate your definition. [4]

The altitude of such a star is 75°23′ at upper transit and 18°09′ at lower transit. Both of these transits occur north of the zenith. Find the latitude of the observing site and the declination of the star. [7]

Calculate the maximum azimuth that the star can have and its altitude and hour angle at that time. [9]

2.32 The Sun is observed to set at 8.00 pm (British Summer Time) at the location of Glasgow (55°52′N, 4°15′W).

(a) What is the Sun's hour angle at this time?

(b) Calculate the Sun's declination.

(c) On the assumption that the obliquity of the ecliptic is 23°26′, calculate the Sun's ecliptic longitude at this time.

(d) What are the two possible dates for this observation?

2.33 In each answer, draw a suitably labelled diagram showing the positions of the transits, the celestial pole and the zenith.

(a) An observer at latitude 48^0N observes a star of declination $\delta = 60^0$N. Prove that this star is circumpolar, and calculate its altitude at upper and lower transit. [8]

(b) Prove that for a circumpolar star of declination δ to have its upper transit south of the zenith for an observer at latitude ϕ, $\phi > \delta$. [5]

(c) A star has zenith distances at lower and upper transits of 24°N of zenith, and 74°S of zenith, respectively. Calculate ϕ and δ. [7]

2.34 Show that the number of hours of daylight on a given day can be estimated as

$$2\cos^{-1}\left(-\tan\delta_\odot \tan\phi_\odot\right).$$

where $\delta_\odot$ is the solar declination, and ϕ is the observer's latitude.

Calculate the hour angle of the setting of the sun for an observer at latitude 54°55′N, given that the sun's ecliptic longitude is 49°49′. What date does this correspond to?

2.35 The star γ Dra has declination 51°29′27". When observed from Strasbourg, it's azimuth ranges between 289°44′32" and 70°15′28". What is the latitude of the observatory at Strasbourg?

3 Dynamical Astronomy

3.1e State Newton's law of gravitation, defining all the symbols you use. Hence derive an expression for the surface gravity of a spherically symmetrical uniform planet in terms of its mass, radius, and the gravitational constant, G. Calculate the value of the surface gravity of Jupiter, which has a mass of 1.9×10^{27} kg and a radius of 7.14×10^7 m. [5]

3.2e Show, assuming Newton's law of gravitation, that the gravitational potential energy, $V(r)$, of a small body of mass m at a distance r from a planet of mass M is given by

$$V(r) = -\frac{GmM}{r},$$

where G is the constant of gravitation. [5]

3.3e State Kepler's Laws of Planetary Motion.

Define the term Angular Momentum and explain, using the case of a circular orbit, the significance of Kepler's Second Law in terms of angular momentum. [5]

3.4e A body of mass m moves under gravity in a circular orbit of radius R about a body of much larger mass M. Given that the centripetal force on a body of mass m moving with velocity v in a circular orbit of radius R is mv^2/R show that the period of the orbit is given by the equation

$$T^2 = \frac{4\pi^2 R^3}{GM},$$ [5]

where G is the constant of gravitation.

3.5 State Newton's law of gravitation. Demonstrate, for a circular orbit, how it is consistent with Kepler's second and third laws of planetary motion. [9]

Show that the orbital period of a mass in a circular orbit of radius r about a much larger mass M is

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM}.$$

What is the corresponding expression for an elliptical orbit? [5]

The comet Faye is in an orbit with a period of 7.38 years, an eccentricity of 0.576 and a perihelion distance of 1.608 AU. The comet Wolf has a period of 8.43 years, and a perihelion distance of 2.507 AU. Calculate the eccentricity of its orbit. {0.395} [6]

3.6e A body of negligible mass m is in an elliptical orbit around the Sun. Write down an expression for the total energy, E, of the body and show that when it is at a distance r from the Sun its velocity, v, can be written as

$$v^2 = GM\left(\frac{2}{r} + C\right),$$

where the constant $C = 2E/(GMm)$ and G is the gravitational constant. [6]

Given that the perihelion and aphelion distances of the orbit are $a(1-e)$ and $a(1+e)$, where a is the semi-major axis and e is the eccentricity of the orbit, use the above expression to find the ratio of the velocity of the body at perihelion to that at aphelion. [2]

From conservation of angular momentum derive a second expression for this ratio of velocities. [2]

Hence show that

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$ [3]

The comet Wolf has an eccentricity of 0.395 and a perihelion distance of 2.507 AU. Calculate the velocity of the comet at perihelion. [4]

3.7e If a body of negligible mass is in orbit around a mass M, its speed, v, is given by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

where r is the distance between the centres of the two masses, a is the semi-major axis of the orbit and G is the gravitational constant. An artificial satellite is in a circular orbit about the Earth at a height of 1 000 km above the surface. Use the above expression to calculate the speed of the satellite. [3]

The satellite is to be transferred to an orbit of radius 30 000 km. Describe an economical way of carrying out this transfer and calculate the change in speed required to inject it into its transfer orbit. [6]

(GM for the Earth $= 3.989 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$. Radius of Earth $= 6\,378$ km.)

3.8e Explain what is meant by the *escape velocity* from the surface of a planet. Given that the gravitational potential energy, V, of a mass m at the surface of a planet of mass M and radius R is $V = -GmM/R$, derive an expression for its escape velocity from the planet's surface. [4]

The orbit of the Moon around the Earth has a semi-major axis of 384 400 km and a period of 27.32 days. Given that the radius of the Earth is 6 378 km, calculate the acceleration due to gravity and the escape velocity at the Earth's surface. [8]

3.9 Derive an expression for the period, T, of a body of negligible mass in a circular orbit with radius R about the Earth. [9]

Derive the *height* above the equator for a geostationary communications satellite, given that the radius of the Earth is 6.38×10^6 m and the (surface) acceleration due to gravity is $9.8\ \mathrm{m\ s^{-2}}$. [8]

How many satellites are needed to give complete coverage of the Earth (excluding polar regions)? [2]

3.10 State Newton's Law of Gravitation and show that the acceleration due to gravity at the surface of a planet of mass M and radius R is given by

$$g = \frac{GM}{R^2},$$

where G is the gravitational constant. [3]

Given that the mass of Venus is 0.817 times that of the Earth and the radius of Venus is 0.97 times that of the Earth, calculate the acceleration due to gravity at the surface of Venus if its value at the surface of the Earth is $9.8\,\mathrm{m\,s^{-2}}$. $\{8.5\,\mathrm{m\,s^{-2}}\}$ [3]

Show that the gravitational potential energy of a mass, m, a distance r from the centre of a planet of mass M ($\gg m$) is

$$V(r) = \frac{-GMm}{r},$$

and write down an expression for the total energy. $\{\frac{1}{2}mv^2 - GMm/r\}$ [8]

Explain the meaning of the phrase 'escape velocity from the surface of a planet' and derive an expression for it in terms of the radius of the planet and the acceleration due to gravity at its surface. Given that the radius of the Earth is 6.38×10^6 m and using the data given and derived earlier calculate the escape velocity from the surface of Venus. $\{10.3\,\mathrm{km\,s^{-1}}\}$ [6]

3.11 By conservation of energy and angular momentum for a small body in an orbit about the Sun we can get the relations

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}, \quad \text{and} \quad h = rv\sin\theta,$$

where M and m are the masses of the Sun and the body respectively, r is the distance of the body from the Sun, G is the gravitational constant, v is the speed of the body, θ is the angle its direction of motion makes with a line to the body from the Sun, and h is a constant.

Explain the origin of the terms in the above equations. [3]

If the orbit is an ellipse of semi-major axis a and eccentricity e, show (by considering the perihelion and aphelion of the orbit) that

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \qquad [10]$$

$$\text{and} \qquad h^2 = GMa(1 - e^2).$$

An asteroid is detected at a distance of 1.26 AU from the Sun. Its velocity is 33.3 km s^{-1} with directed at 74.7° to the line joining the two. Calculate the semi-major axis and period of its orbit. Does the orbit cross that of the Earth? [7]

(Use Solar units for this problem, remembering that 1 km s^{-1} = 0.211 AU yr^{-1}.)

3.12 Show that the laboratory value of the gravitational constant can, with other measurements given below, give the masses of the Sun and the Earth.

semi-major axis of Earth's orbit	1.496×10^8 km
radius of Earth	6371 km
surface gravitational acceleration on Earth	9.81 m s^{-2}
Earth year	365.25 d

3.13 Assuming the orbit of Mars to be circular and in the ecliptic with a synodic period of 780 days calculate (without using Kepler's third law):

(a) the orbital period in days

(b) the radius of the orbit in astronomical units, given that, 35.61 days after opposition, the elongation is 139°05′

(c) the interval between opposition and the next quadrature.

3.14 The satellite *Europa* describes an orbit about Jupiter of semi-major axis 6.71×10^5 km in a period of 3.552 days. Neptune's satellite *Triton* has semi-major axis 3.55×10^5 km and orbital period 5.877 days. Calculate the ratio of the masses of the two planets. {18.49}

3.15 Halley's comet has an orbital period of 76 years and its perihelion distance is 0.59 AU. Calculate its semi-major axis and its greatest distance from the Sun in AU. Use your lecture notes to calculate its velocity at perihelion and aphelion in AU yr^{-1}. Convert these to km s^{-1}, given that the Earth's orbital velocity is 29.8 km s^{-1}. What is the ratio of its greatest and least orbital velocities? {17.942 AU; 35.294 AU; 11.47 AU yr^{-1}; 0.192 AU yr^{-1}; 59.7}

3.16 A comet, moving towards perihelion, has a velocity of 31.53 km s^{-1} when 1.70 AU from the Sun, directed at 143.16° from the direction of the radius vector. Use this information to calculate a and h. Also obtain the eccentricity, perihelion and aphelion distances and period of the orbit. {a = 17.84 AU; e = 0.966; 0.607 AU; 35.07 AU; 75.35 yr}

3.17 The satellite *Phobos* of Mars has an orbit with a period of 0.318 9 days and a semi-major axis of 9.38×10^3 km. The diameter of Mars is 6 762 km. Show that, without a knowledge of the gravitational constant, this is sufficient to calculate the surface gravity of Mars and its surface escape velocity. Obtain values for these quantities.

3.18 Explain what is meant by a Hohmann transfer orbit.

A manned spacecraft leaves the vicinity of the Earth to go to Venus using the most economical transfer orbit. Calculate the minimum change in velocity required to reach the orbit of Venus and the time taken for the transfer. If the spacecraft is to rendezvous with Venus, what should be the position of Venus relative to the Earth in order to achieve this? Calculate the change in velocity required once the spacecraft reaches Venus. How long is required before the spacecraft can return to Earth using a Hohmann transfer orbit?

Assume that the orbits of Earth and Venus are circular and coplanar with radii of 1.0 and 0.723 AU respectively. Ignore the gravitational effects on the spacecraft by Earth and Venus.

3.19 The semi-major axis of the orbit of Mars is 1.524 AU and the orbital eccentricity is 0.093. Assuming the Earth's orbit to be circular and coplanar with that of Mars, calculate:

(a) the distance of Mars from the Earth at closest approach
(b) the ratio of the speeds of Mars in its orbit at perihelion and aphelion
(c) the speed of Mars at aphelion in AU per year.

3.20 Two artificial satellites are in elliptical orbits about the Earth and both have the same period. The ratio of the velocities at perigee is 1.5 and the eccentricity of the satellite with the greater perigee velocity is 0.5. Calculate the eccentricity of the orbit of the other satellite and the ratio of the apogee velocities of the two satellites.

3.21 Halley's Comet moves in an elliptical orbit with an eccentricity of 0.967 3. Calculate the ratios of

(a) the linear velocities
(b) the angular velocities at aphelion and perihelion.

3.22 The period of Jupiter is 11.86 years and the masses of Jupiter and the Sun are respectively 3.3×10^5 and 318 times that of the Earth. Calculate the change in Jupiter's orbital period if the semi-major axis was the same but its mass was the same as the Earth.

3.23 A lunar probe is put into an elliptical transfer orbit from a circular parking orbit (radius 6 878 km) about the Earth. It is intended that the apogee of the transfer orbit should touch the Moon's orbit (assumed circular with a radius of 3.844×10^5 km). If the velocity in the parking orbit is 7.613 km s^{-1}, calculate:

(a) the semi-major axis and eccentricity of the transfer orbit
(b) the time the probe takes to reach apogee
(c) the required velocity increment to give the transfer orbit.

3.24 Explain what is meant by the linear momentum of a body and describe how it changes when a force is applied to the body.

(a) Explain what is meant by the angular momentum of a body about an axis. What quantity (analogous to force) changes the angular momentum of a body?

(b) Write down Kepler's laws of planetary motion. Explain, using a circular orbit, the physical basis of the second law.

(c) Show how a cone can be sliced to give a circle, an ellipse or a hyperbola. Where does a parabola fit in here?

(d) Consider a satellite in a circular orbit about the Earth. At a particular point in the orbit the spacecraft engines are fired to increase the tangential velocity of the craft. Sketch the possible trajectories of the spacecraft.

(e) Sketch an ellipse and mark clearly the positions of the foci. If the Sun is at one focus of an elliptical orbit, mark the positions of perihelion and aphelion and write down expressions for the perihelion and aphelion distances in terms of the semi-major axis and eccentricity of the ellipse.

The comet Encke has an orbit with a period of 3.3 years, a perihelion distance of 0.339 AU and an eccentricity of 0.847. Comet Halley has an orbit with perihelion distance of 0.587 AU and eccentricity of 0.967 AU. Calculate the length of the semi-major axis for each of the above orbits and using Kepler's third law find the period of comet Halley.

3.25e State Kepler's three laws of planetary motion.

Given that the semi-major axis of the orbit of Venus is 0.723 3 AU, calculate its sidereal period in years. [5]

3.26e The orbital period, T, of a body of mass m in an elliptic orbit about another body of mass M is

$$T = 2\pi \left[\frac{a^3}{G(M+m)}\right]^{1/2} = 2\pi \left(\frac{a^3}{\mu}\right)^{1/2} ,$$

where a is the orbital semi-major axis and G the gravitaional constant. Also if v is the speed of the body of mass m when its radius vector is r, then

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right) .$$

Show that, if at any time the direction of motion of m is changed without changing the magnitude of the velocity and without changing the length of the radius vector r, the semi-major axis a and the period T of the resulting orbit remain unaltered. [5]

3.27e A planet of mass m is in orbit about the Sun, of mass M. The sum of the kinetic and potential energy of the planet in its orbit is a constant, C, given by

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C ,$$

where v is the magnitude of the velocity of the planet when its radius vector is r, $\mu = G(M+m)$ and G is the gravitational constant. Show that

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right) ,$$

where a is the planet's orbital semi-major axis. [11]

Halley's Comet moves in an elliptical orbit of eccentricity 0.9673. Calculate the ratio of the magnitudes of (a) its linear velocities, (b) its angular velocities, at perihelion and aphelion. [6]

3.28e Two planets P_1 and P_2 of masses m_1 and m_2, orbital semi-major axes a_1 and a_2, and orbital periods T_1 and T_2 respectively, are in orbits about a star of mass M. Using the expression

$$T = 2\pi \left(\frac{a^3}{\mu}\right)^{1/2} ,$$

where $\mu = G(M+m)$ and G is the gravitational constant, derive Newton's form of Kepler's third law. [8]

A satellite of Jupiter has an orbital period of 0.498 2 days and an orbital semi-major axis about Jupiter of 0.001 207 AU. Jupiter's orbital period and semi-major axis about the Sun are 11.86 years and 5.203 AU respectively. Calculate the ratio of the mass of Jupiter to that of the Sun. [9]

3.29e Explain the importance of conservations laws for dynamical astronomy calculations, and give two examples of quantities that are conserved during simple gravitational interactions. [5]

A body of mass m is in orbit about a planet of mass $M (\gg m)$ taken to be at rest. Write down an expression of the total energy of this system in terms of the body's speed, v, and distance from the centre of the planet, r, and show that

$$v^2 = GM\left(\frac{2}{r} + C\right)$$

where C is a constant for the orbit. [4]

It has been suggested that liquid oxygen fuel pods, mined from the Moon's South Pole, could be launched to waiting spacecraft in orbit at a radius of 2 lunar radii about the Moon. Given that $C = -1/a$, where a is the semi-major axis of the orbit, use the above equation to determine the initial velocity required

(a) launching the pods vertically from the surface, [3]

(b) launching the pods parallel to the surface. [3]

What other major factor would be important to complete the transfer? Explain why (b) might be preferred overall. [2]

4 Solar System Physics

4.1e List five main differences between the terrestrial and jovian planets. [5]

4.2e Give a sketch of the Earth's interior, clearly indicating the different regions, and their approximate dimensions. [5]

4.3e Explain the terms

(a) plate tectonics

(b) volcanism

(c) outgassing. [5]

4.4e Give a sketch of the atmosphere of the Sun, indicating the approximate dimensions of the various regions and their approximate temperatures. [5]

4.5e The Earth and Jupiter's satellite Io both exhibit volcanism. Briefly describe the causes of this volcanism in each case. [5]

4.6e Sketch the *internal* structure of the Sun, indicating the approximate dimensions and temperatures of the various regions. [5]

4.7e Explain the mechanism of the greenhouse effect. Give the names of two main greenhouse gases in the Earth's atmosphere. [5]

4.8e In what way do the surface features of the Moon and Earth radically differ and why? [5]

4.9e The surface temperature of Venus is 700 K and that of the Earth is 300 K. Calculate the wavelengths at which the radiative flux from each planet is maximum. You may use Wien's law, $\lambda_{\max} T = 2.9 \times 10^{-3}\,\mathrm{m\,K}$. [2]

In what region of the spectrum do these wavelengths lie? [1]

Explain carefully what is meant by the 'greenhouse effect' in planetary atmospheres. Discuss this with reference to the atmospheres of the terrestrial planets. [10]

How does the 'runaway' greenhouse effect explain the near absence of water in Venus's atmosphere? [4]

4.10e Explain briefly what evidence there is to support the statement that the Moon's age is about 5 000 million years? [4]

The abundance of the radioactive isotope of potassium, ${}^{40}_{19}\mathrm{K}$, found in a sample of rock found on the Moon is found to be 1 part in 10^8, whereas the abundance of argon, ${}^{40}_{18}\mathrm{Ar}$, is found to be 2 parts in 10^9. Assuming that all the argon was produced through the radioactive decay of ${}^{40}_{19}\mathrm{K}$, estimate the age of the sample of rock and comment on the age you obtain. [8]

Explain why the rate ratio of radiogenic heating to the rate of solar heating would be greater in the early life of a planet. Estimate the percentage change in the rate of radiogenic heating due to ${}^{40}_{19}\mathrm{K}$ decay over the Moon's lifetime. [5]

(You may assume that the number, N, of parent radioactive isotopes obeys the exponential decay law, $N = N_0 \exp(-\lambda t)$. Take the half-life of ${}^{40}_{19}\mathrm{K}$ to be 1.3×10^9 years.)

4.11e Show that the surface temperature, T_p, of a planet is given by

$$T_\mathrm{p} = (1 - A)^{1/4} \left(\frac{R_\odot}{2r} \right)^{1/2} T_\mathrm{e},$$

where r is the distance of the planet from the Sun, and $R_\odot$ is the solar radius, T_e is the effective temperature of the Sun, and A is the albedo of the planet. State clearly any assumptions that you make. [10]

Assuming that Mars has an albedo of 0.2, calculate its surface temperature given that the orbital period of Mars is 1.9 years. [5]

What evidence is there for the statement that the surface temperature was considerably higher than this in the past? [2]

(For the Sun, $T_e = 5\,800$ K.)

4.12 State the main differences between the properties of the terrestrial planets and the jovian planets. [6]

Show that the escape velocity is given by $v_e = (2GM/R)^{1/2}$.(You may assume that the gravitational potential energy of body of mass m on the surface of a sphere of mass M and radius R is given by $-GMm/R$ and its kinetic energy is given by $mv^2/2$.) Calculate the ratio of the escape velocities for the two planets given the following information: [10]

	Venus	Jupiter
Mass in kilograms	4.87×10^{24}	1.90×10^{27}
Radius in metres	6.05×10^{6}	7.14×10^{7}

How does the difference in escape velocities explain the difference between the chemical composition of the atmospheres of the two planets? {5.85} [4]

4.13 Sketch the monochromatic energy flux, F_λ, against wavelength, λ, at the Sun's surface, clearly indicating the wavelength at which the flux is a maximum. You may assume Wien's law. [4]

Show that the total radiative flux from the sun falling on a planet's surface is given by

$$\left(\frac{R}{r}\right)^2 \sigma T_e^4,$$

where r is the distance of the planet from the Sun, R the radius of the Sun, and T_e the effective surface temperature of the Sun.

Assuming that a planet is in radiative equilibrium, and that a fraction A of the Sun's radiation is reflected by the planetary surface, show that the surface temperature of the planet is given by

$$T_p = (1 - A)^{1/4} \left(\frac{R}{2r}\right)^{1/2} T_e.$$ [12]

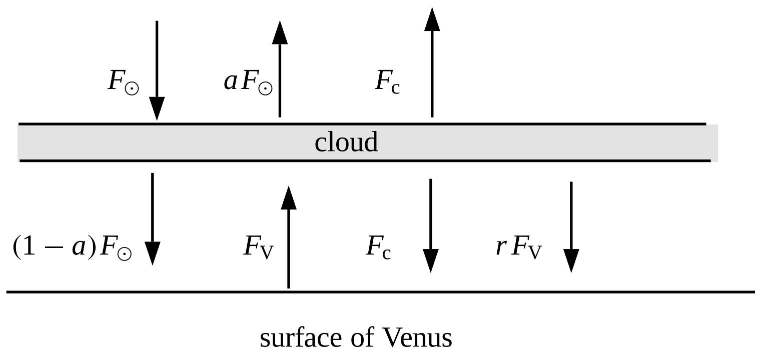
Calculate the surface temperature for Mars given that its albedo, A, is 0.16 and that it is at a mean distance of 1.52 AU from the Sun. {217.2 K} [3]

At what wavelength does Mars emit most of its radiation? [3]

(You may assume that $T_e = 5\,800$ K, and 1 AU= 1.496×10^{11} m and that the radius of the Sun is 6.960×10^8 m.)

4.14 If one ignores the effect of Venus's atmosphere the predicted surface temperature would be around 240 K, which is the temperature above the highly reflective cloud. The measured surface temperature however is about 700 K. Explain the reasons for this difference. [6]

A simple model of the atmosphere of Venus is described by the figure below.



$F_\odot$ is the radiative flux from the Sun arriving at the top of Venus's atmosphere. The radiative flux from the Sun reflected by the cloud is given by $aF_\odot$ where a is the albedo, and the radiative flux transmitted through the cloud is $(1-a)F_\odot$.

The radiative flux emitted by the upper and lower surface of the cloud is given by F_c. A fraction r of the radiative flux, F_V, emitted from Venus's surface is reflected by the cloud and the remaining fraction $(1-r)$ is absorbed by the cloud. Assuming the radiative flux is in overall balance, show that $F_c = (1-a)F_\odot$ and $F_V = F_\odot(1-a) + rF_V + F_c$. Hence show that

$$\frac{F_V}{F_c} = \frac{2}{1-r}$$ [8]

Assuming that both cloud and Venus's surface radiate as black bodies, and that the cloud is at a temperature of 240 K, calculate the value of r necessary to yield the measured surface temperature of Venus of 720 K. {0.975} [6]

4.15 What is meant by the term *greenhouse effect*? Explain why it is particularly important for Venus but not for Mars. [5]

The surface atmospheric pressure of Mars is 0.006 times that of the Earth. Taking the temperature of the lower atmosphere of Mars to be 250 K calculate the density of its atmosphere. (1 atmosphere is approximately $10^5\,\mathrm{N\,m^{-2}}$.) Assume for the purposes of this question that Mars' atmosphere is 100 % CO_2. The mass number of atomic oxygen, O, is 16 and C is 12. {$0.0128\,\mathrm{kg\,m^{-3}}$} [8]

What is meant by the atmosphere being isothermal? Assuming this to be the case calculate the scale-height of Mars' atmosphere, and sketch the atmospheric pressure as a function of height. {12.581 km} [5]

At what height is the pressure 0.002 Earth atmospheres? {13.81 km} [2]

(Mass of Mars = $6.42 \times 10^{23}\,\mathrm{kg\,m^{-3}}$, radius of Mars = 3.39×10^6 m.)

4.16 Explain the terms

(a) igneous,

(b) sedimentary,

(c) metamorphic, and

(d) primitive rocks.

Where might these rocks be found? [8]

Calculate the age of a rock sample which contains 46 parts per billion of ^{40}K and 12 parts per billion of ^{40}Ar. Assume that all of the ^{40}Ar found is the direct result of decay from ^{40}K, with a half-life of 1.3×10^9 years. {4.3×10^8 yr} [8]

Explain why this rock would be considered relatively ancient if found on the Earth's surface but not if it were found on the Moon. [4]

4.17 Discuss briefly the internal structure and composition of the *terrestrial planets*, pointing out the similarities and differences. [8]

Explain the term *radiogenic heating*. [2]

The decay of ^{40}K into ^{40}Ar is responsible for about 50 % of the radiogenic heating of the Earth today [^{232}Th(orium), ^{238}U(ranium) and ^{235}U provide the rest]. 1 kg of Earth contains about 10^{-8} kg of ^{40}K. The atomic mass of ^{40}K is 39.97 amu and the atomic mass of ^{40}Ar is 39.96 amu. The half life of ^{40}K is about 109 years. *Estimate* the amount of heat thus generated per year per kg of Earth material, and hence *estimate* the power in watts due to radiogenic heating for the whole Earth. {$\sim 2 \times 10^{-4}$ J yr^{-1}; $\sim 3 \times 10^{13}$ W} [8]

How does this compare with the rate of heating due to the Sun? {$\sim 10^{17}$ W} [2]

(You may assume that the mass of the Earth is about 6×10^{24} kg and the solar constant is 1.4×10^3 W m^{-2}.)

4.18 Explain what is meant by a tidal force and explain in qualitative terms the origin and meaning of the Roche stability limit. [8]

The Roche stability limit of a planet is approximately given by 2.5 times the planetary radius. If the radius of Jupiter is 71 000 km and Saturn 60 000 km, estimate the likely maximum distance that a ring could be found from the centre of either planet. {1.8×10^5 km; 1.5×10^5 km} [4]

Give a possible explanation for the formation and the structure of these rings. [4]

Describe the asteroid belt, and give a plausible explanation for the Kirkwood gaps. [4]

4.19 Outline the currently accepted view of how the solar system formed from interstellar material. Describe the main features of the solar system that this model explains (its structure, chemical composition, age and orbits/motion of planets). [20]

[Your answer should be succinct, factual, and about 300 words long (no more than 400). It can be in note form.]

4.20 Discuss what is meant by *Bode's Law* and briefly describe its significance in relation to a physical understanding of why the planets are found at their respective distances from the Sun.

Another expression relating the planetary mean distances, r, from the Sun (in AU) according to their number order, n, is $n = 3 + 4.11 \log_{10} r$. Using the tabulated information given in your lecture notes, how well do Venus, Earth and Jupiter fit the equation?

4.21 From the data given in your lecture notes, calculate the average density of each of the planets compared to the Earth. Group the results according to any broad features you observe.

4.22 Show that the tidal force acting on a body, mass m, radius R, when at a distance r from an object of mass M, is given by

$$F_{\text{tidal}} = \frac{4GMmR}{r^3},$$

by considering the difference in gravitational forces exerted on the far and near side of the affected body. Assume $r \gg R$.

Calculate the ratio of the tidal forces exerted by Jupiter on its satellites *Io* and *Callisto*. Explain how the difference in tidal forces between these two satellites is reflected in their physical properties. ($r_{\text{Io}} = 4.22 \times 10^5$ km, $r_{\text{Callisto}} = 1.88 \times 10^6$ km, $m_{\text{Io}} = 8.92 \times 10^{22}$ kg, $m_{\text{Call}} = 1.08 \times 10^{23}$ kg, $D_{\text{Io}} = 3630$ km, $D_{\text{Callisto}} = 4\,800$ km) {55.2}

4.23 Calculate the ratio of the surface gravities of the Moon and Mercury. Explain the consequences of the answer in terms of the impact cratering on each body. ($M_{\text{Moon}} = 0.012 M_{\text{Earth}}$, $M_{\text{Mercury}} = 0.055\,8 M_{\text{Earth}}$, $D_{\text{Moon}} = 0.27 D_{\text{Earth}}$, $D_{\text{Mercury}} = 0.381 D_{\text{Earth}}$) {0.43}

4.24 Compare the surface gravity of Jupiter and Saturn. By considering the lower atmospheres of each planet, estimate the altitude difference between the formation of NH_3 and H_20 clouds for Jupiter and Saturn. Is this consistent with a scale-length $h = kT/(mg)$ for atmospheric variations? Use your notes to get the atmospheric data. ($M_{\text{Jup}} = 318 M_{\text{Earth}}$, $M_{\text{Sat}} = 95.2 M_{\text{Earth}}$, $D_{\text{Jup}} = 10.86 D_{\text{Earth}}$, $D_{\text{Sat}} = 9 D_{\text{Earth}}$)

4.25 Calculate the age of a rock sample which contains 46 parts per billion of ^{40}K and 12 parts per billion of ^{40}Ar. Assume that all of the ^{40}Ar found is the direct result of decay from ^{40}K, with a half-life of 1.3 billion years.

Would this rock be considered relatively ancient if found on the Earth's surface? What if it were a Moon rock? {4.35×10^8 yr}

4.26 The angular diameter of the Sun as seen from Earth is 32 arcmin. Calculate the radius of the Sun. Assuming the Sun emitted its luminosity like a perfect black body, calculate its effective temperature, and compare it with Earth's surface temperature.

4.27 In association with the apparent brightness of an asteroid, define the term *absolute magnitude*. Demonstrate how absolute magnitude depends on the physical size of the body and on its albedo. Show that the apparent magnitude, m, of an asteroid may be expressed by

$$m = m_0 + 5\log_{10} r + 5\log_{10} D,$$

where m_0 is the absolute magnitude of the asteroid, r its distance from the Sun and D its distance from the Earth. The asteroids *Ceres* and *Vesta* have had their diameters measured directly as 1 000 km and 540 km respectively. At opposition their magnitudes are 6.7 and 5.3 and their solar distances 2.77 and 2.36 AU. Calculate the ratio of their albedos. {0.187}

4.28 A spherical asteroid has a mean absolute magnitude of 10 and is observed at a heliocentric distance of 1.5 AU and at an Earth distance of 1.2 AU. It rotates about an axis perpendicular to its orbital plane which is coplanar to the Earth's orbital plane. During its rotation, it alternately presents two distinct hemispheres, one with an albedo of 0.10, the other with 0.05. Sketch out the light curve (magnitude against time – quantitatively) for the object.

4.29 Explain how measurements of the polarization of the light from an asteroid over a range of phase angles can lead to an estimation of its albedo and hence to a determination of its diameter. Two asteroids with orbital radii 2.2 and 2.5 AU have the same apparent magnitude at opposition. The respective values of the polarization at the turning point, $P_{\rm min}$, on the $P(q)$ curve are 1 % and 0.5 % respectively. Determine the ratio of their radii.

4.30 The equation for hydrostatic equilibrium is $dp/dz = -g\rho$ where z is the height above the surface, g the surface gravitational acceleration, p the pressure and ρ the gas density. Using the ideal gas law $p = \rho kT/\mu$, where T is the temperature, k is Boltzmann's constant and μ the average mass of a gas particle, show that in an isothermal atmosphere $p = p_0 \exp(-z/H)$, where $H = kT_0/(\mu g)$. We think of H as the scale-height for variations in the atmosphere. What could the pressure profile be if the atmosphere was not isothermal?

4.31 Suppose the surface of a sphere the same size as the Sun was covered in electric light bulbs. What would the wattage of each bulb have to be in order to match the Sun's luminosity? Assume the surface area of a bulb is 30 $\rm cm^2$.

4.32 Compare the tidal force exerted by the Sun on the Earth with that exerted by the Moon on the Earth. Explain your answer in terms of the spring and neap tides.

4.33 The Stefan-Boltzmann Law is $L = 4\pi R^2 \sigma T^4$, and Wien's Law is $\lambda = W/T$. Describe how these formulae can be applied to planetary energy budgets, explaining what each symbol means.

Mars has an orbital radius about the Sun of $a = 1.524$ AU, and a planetary radius of 3.38×10^3 km. Ignoring Mars's atmosphere, calculate:

(a) the power supplied by the Sun to Mars,

(b) the power radiated by Mars assuming the planet has an albedo of 0.15,

(c) the dominant wavelength of such emissions from Mars.

Given that observations show that Mars radiates predominantly at a wavelength of 1.45×10^{-5} m, compare this with your answer for the dominant wavelength, and account for any discrepancy. ($W = 2.90 \times 10^{-3}$ km)

4.34 State Wien's Law, and describe briefly how it may be used. What is the effective temperature of an object? Radiation from Mars is observed predominantly at a wavelength of $\lambda = 1.45 \times 10^{-5}$ m. Calculate the surface temperature and hence the implied luminosity of Mars. The effective temperature of Mars is 217 K. Calculate the intrinsic luminosity of Mars using this temperature, and compare it with your earlier answer. Can you explain the difference?

Given that Mars has a mean orbital radius $a = 2.3 \times 10^{11}$ m, calculate its albedo. (Planetary radius of Mars $R_{\rm M} = 3.4 \times 10^6$ m, $W = 2.9 \times 10^{-3}$ K m)

4.35 Give a sketch of the main regions of the solar atmosphere, indicating the temperature and dimensions.

Using Wien's displacement law calculate the typical wavelength and energy in eV of the photons emitted from these regions, and state the region of the spectrum in which they lie.

The ionisation potential of atomic hydrogen is 13.6 eV. What would the frequency of a photon have to be in order to ionise atomic hydrogen? What would you expect the ionisation potential of ${}^{4}_{2}$He and ${}^{56}_{26}$Fe to be? (Hint: the ionisation potential of the last electron in an atom is proportional to Z^2, where Z is the atomic number.) What would the corresponding frequency/energy of a photon have to be?

The Sun's luminosity comes from hydrogen fusion in the core of the Sun, which essentially converts 4 protons into ${}^{4}_{2}$He. Given the mass of a proton is 1.0073 amu, and the mass of ${}^{4}_{2}$He is 4.0026 amu, calculate the *fraction* of the mass of a proton that gets converted into energy.

Estimate the relative efficiency of a hydrogen bomb to TNT or some other chemical explosive.

How many kg of H need to be converted into ${}^{4}_{2}$He per second to provide the luminosity of the Sun?

(Subversive question: how much hydrogen would be necessary to blow the Houses of Parliament 100 metres into the sky?)

(Hysterical question: How much Hydrogen would be necessary to deviate a 1 000 m diameter asteroid from collision course with Earth?)

4.36 What are meant by the terms

(a) plate tectonics
(b) volcanism
(c) outgassing?

If the oceans are assumed to have an average depth of 4 km and to cover 2/3 of the Earth's surface calculate the mass of water they contain. The present rate of outgassing of water is measured to be 2×10^{10} kg per year. Assuming that this has been constant throughout the lifetime of the Earth, calculate the age of the Earth.

By considering hydrostatic equilibrium, show that the pressure at the bottom of the ocean must be given by $Mg + P_a$, where M is the mass of water per unit area, g is the surface gravity of the Earth and P_a is the Earth's atmospheric pressure at sea level.

If all the water were evaporated from the oceans, what would be the pressure of the Earth's atmosphere? How does this compare with Venus's atmosphere?

Mars, Earth and Venus might have had similar atmospheres in the early solar system. In what respects do they differ? What explanations do you have for this?

Radius of the Earth is 6 378 km, 1 atmosphere is 10^5 N m^{-2}, density of water is 10^3 kg m^{-3} and $g = 9.8$ m s^{-2}.

4.37 Calculate the surface gravity, g_V, of Venus given that its radius is 0.95 times the radius of the Earth, and its mass is 0.82 times the Earth's mass. (You may assume the surface gravity of the Earth is approximately 9.8 m s^{-1}.)

(a) Show that the mass of atmosphere per square metre of Venus's surface is given by $\frac{P_V}{g_V}$ where P_V is the surface pressure of Venus.

(b) Hence estimate the mass of Venus's atmosphere given that the surface pressure of Venus is 90 times that of the Earth, and 1 atmosphere is about $10^5\,\mathrm{N\,m^{-2}}$. (You may assume the radius of the Earth.)

(c) What is the composition of Venus's atmosphere?

(d) What is Venus's rotation period?

(e) The surface temperature of Venus is about 700 K. This surface temperature is fairly uniform over the surface of Venus. What explanations do you have, bearing in mind Venus's rotation period?

(f) Estimate the density of the atmosphere at the surface of Venus. (You may assume the value of the Boltzmann constant, k.)

(g) How do you expect this density to vary with height?

Other possible discussion points.
Surface gravities of other planets. (Jovian planets have maintained their light elements.)
Greenhouse effect – what is it?
Why is there so little water on Venus?
Why is there so little CO_2 in Earth's atmosphere?
Isothermal atmosphere exponential behaviour of pressure and density.
Convective atmosphere.

4.38 The gravitational force produced by a body of mass M on another of mass m is given by $-GMm/r^2$.

(a) (i) Use this result to show that the surface gravity at the Earth's surface is $-GM_\oplus/R_\oplus^2$. The Earth is made up of a metallic core of iron and nickel (about 95% iron). The density of iron at atmospheric pressure and 300 K is about $8\,000\,\mathrm{kg\,m^{-3}}$, and nickel a bit more, about $9\,000\,\mathrm{kg\,m^{-3}}$. The mantle is molten lava with density about $5\,200\,\mathrm{kg\,m^{-3}}$, and the crust is a thin layer (about 100 km or so) of rock with a somewhat lower density.

(ii) Assuming that the core has a radius of 2 000 km, estimate the total mass of the Earth, and its mean density assuming that the radius is 6 400 km.

(iii) Given that the measured surface gravity of the Earth is $9.8\,\mathrm{m\,s^{-2}}$, estimate the value of G.

(iv) If the core radius was 2 500 km, what value of G would you obtain?

(b) (i) Calculate the relative size of the gravitational forces exerted by the Earth and the Sun on the Moon.

(ii) Calculate the relative size of the gravitational forces exerted by the Moon and the Sun on the Earth.

(iii) Tidal forces are proportional to M/r^3. Estimate the relative size of the tidal force of the Sun and the Moon at the surface of the Earth.

(iv) Discuss the relevance of this result for spring and neap tides.

In part (b) you may assume the mass of the Earth, the Sun and the Moon. You may also assume that the Moon-Earth distance is 1.2 light seconds, and that the Earth-Sun distance is 8 light minutes.

4.39e Briefly discuss the *runaway greenhouse effect*, and how it can account for the lack of water on Venus. [5]

4.40e Describe what is meant by the process of *differentiation*, and comment on its effect on the general structure of the interior of the terrestrial planets. The planet Mars has a reddish appearance due to significant amounts of iron oxide on its surface. Comment on what this tells you about the differentiation process on Mars. [5]

4.41e Rocks on Earth may be classified into three main types according to how they were formed. Name two of these types and describe the processes by which they have been formed.

A further class of rocks are the *primitive* rocks. Where are these typically found? [5]

4.42e Briefly describe with the aid of a diagram, the interior structure of Jupiter. What are the two dominant elements in Jupiter's atmosphere? [5]

4.43e Using Newton's Law of Gravitation, write down the force acting on a point mass m at position X which is a distance r from a body of mass M, and give its direction. Also write down the force on such a mass m if it were at position Y, a distance $r + a$ from mass M, where X and Y lie along the same radius vector. Assuming $a \ll r$, show that the magnitude of the force on m at position Y can be approximated to

$$F_{\mathrm{Y}} = \frac{GMm}{r^2}\left(1 - \frac{2a}{r}\right) .$$

[You may use the binomial theorem:$(1 + x)^n \approx 1 + nx$, for $|x| \ll 1$.]

Hence show that the magnitude of the tidal force between two masses, each of mass m, at these positions is given by $2GMma/r^3$. [5]

Explain with the aid of a diagram how the Moon produces tides on the Earth, and why there are two high tides per day. [4]

Calculate the *ratio* of the tidal force exerted by the Sun at the Earth to that exerted by the Moon at the Earth. [4]

Explain with reference to the positions of the Sun and the Moon why spring tides are stronger than neap tides. [4]

[mass of Sun = 1.99×10^{30} kg,
mass of Moon = 7.35×10^{22} kg,
mean distance of Earth from Sun = 1.50×10^{11} m,
mean distance of Moon from Earth = 3.84×10^{8} m.]

4.44e The dating of rocks can be carried out using the law of radioactive decay of unstable isotopes, given by

$$N = N_0 \exp(-\lambda t) ,$$

where λ is the decay constant. What are N and N_0 in this equation? What is meant by the half-life, $t_{1/2}$, for a radioactive isotope? Sketch a graph of N versus t, clearly marking the position of $t_{1/2}$.

Show that $t_{1/2} = (\log_e 2)/\lambda$. [8]

A rock sample is found to contain 5 parts per billion of the radioactive isotope of potassium, ^{40}K, and 35 parts per billion of argon, ^{40}Ar. Assuming all of the ^{40}Ar found is the direct result of decay from ^{40}K, calculate the age of the rock. The half-life for decay of ^{40}K is 1.3×10^9 years.

If this rock were found on Earth comment on whether its age would be typical of Earth rocks. [5]

Heating by radioactive decay of naturally occurring unstable isotopes is believed to be the main source of energy within the Earth which drives plate tectonic activity. Give the name of one other radioactive element, apart from potassium, which contributes to this heating, and briefly explain what the term *plate tectonics* means. [4]

4.45 (a) What is meant by the effective temperature of a stellar body? [1]

The effective temperature of the Sun is 5 800 K. Calculate the flux at the surface of the Sun, and give the units of flux. [3]

Hence calculate the luminosity of the Sun. [2]

The monochromatic flux F_λ from the Sun peaks in the visible region. At what colour is the peak? [1]

Saturn is approximately 10 times further away from the Sun than the Earth is from the Sun. Give the approximate value for the ratio of the Sun's flux falling on Saturn to that falling on the Earth. Justify your answer. [3]

(b) The ideal gas law may be expressed as $P = \frac{\rho k T}{\mu m_{\rm H}}$. Give the meaning of all the symbols used in this equation. [3]

The atmospheric pressure at Venus's surface is $90 \times 10^5\,{\rm N\,m^{-2}}$. Comment on how this compares to the atmospheric pressure at the Earth's surface. [1]

Assuming that the surface temperature of Venus= 740 K, and that the atmosphere is entirely made of carbon dioxide (CO_2), calculate the surface density of Venus's atmosphere. [2]

Calculate the scale height for CO_2 on Venus, and the pressure at this height, assuming an isothermal atmosphere. [4]

[Assume $\mu = 12 + (2 \times 16)$ for CO_2
Surface gravity of Venus $= 8.88\,{\rm m\,s^{-2}}$
$m_H = 1.67 \times 10^{-27}$ kg.]

4.46 (a) Give the equation for the magnitude of the gravitational force produced by a body of mass M on another mass m separated by a distance r. In what direction does the force act?

Hence show that the surface gravity on a planet of mass M and radius R is given by $g = GM/R^2$. [4]

A measurement of surface gravity on the Moon gives a value of $g_{\rm Moon} = 1.6\,{\rm m\,s^{-2}}$.

Calculate the mass of the Moon. [3]

Calculate the ratio of the surface gravity on Jupiter to that on Pluto. [3]

[radius of Moon= 1.74×10^6 m
mass of Jupiter= 1.90×10^{27} kg, radius of Jupiter= 7.14×10^7 m
mass of Pluto= 1.29×10^{22} kg, radius of Pluto= 1.20×10^6 m.]

(b) Explain with the aid of a diagram how the Moon produces tides on the Earth and why there are two high tides per day. [4]

Tidal forces are proportional to M/r^3 where M is the mass of the body causing the force and r is the average distance from M. Calculate the relative size of the tidal force exerted by the Sun at the Earth to that exerted by the Moon at the Earth. [3]

Explain with reference to the positions of the Sun and the Moon why spring tides are particularly strong. [3]

[mass of Sun$= 1.99 \times 10^{30}$ kg, mass of Moon$= 7.35 \times 10^{22}$ kg
mean distance of Earth from Sun$= 1.50 \times 10^{11}$ m
mean distance of Moon from Earth$= 3.84 \times 10^{8}$ m.]

4.47 (a) Under the condition of hydrostatic equilibrium, and by considering the forces acting on a column of gas, show that the mass of atmosphere per unit area of surface of a planet is given by P/g, where P is the atmospheric pressure at the surface and g is the surface gravity. Draw a suitable diagram to illustrate your solution. What assumption are you making in taking a constant value of g through the atmosphere? [5]

Calculate the mass of atmosphere on Mars per square metre of surface. Hence calculate the total mass of Mars's atmosphere. What is the principle gas in Mars's atmosphere? [6]

[Atmospheric pressure at surface of Mars$= 6.0 \times 10^{2}\,\mathrm{N\,m^{-2}}$
Surface gravity on Mars$= 3.7\,\mathrm{m\,s^{-2}}$ Radius of Mars$= 3400$ km.]

(b) The Earth has an average density of $5.5\times10^{3}\,\mathrm{kg\,m^{-3}}$ and a radius of 6 400 km. Calculate the total mass of the Earth. [3]

The interior of the Earth consists of a dense core, surrounded by a mantle and a crust. If the core has a radius of 3500 km and an average density of $12 \times 10^{3}\,\mathrm{kg\,m^{-3}}$, what is the mass of the core? Hence calculate the percentage of the mass of the Earth which is contained in its core. [3]

All the terrestrial planets are believed to contain dense cores. Name the process by which such dense cores have been formed. [1]

The core of the Earth is partially in liquid state. Which other terrestrial planet is believed to have a partially liquid core and what evidence do we use to infer this? [2]

4.48 The empirical Titius-Bode Law for the distance a_n of the n-th planet from the sun can be written

$$a_n = \begin{cases} 0.4 & n = 1 \\ 0.4 + 0.3 \times 2^{n-2} & n \geq 2 \end{cases}$$

Draw a graph of this function a_n against n, and superpose on it points showing the actual distance of the 9 planets, omitting $n = 5$, which is the asteroid belt, taking $n = 6$ for Jupiter and so on outward. Use a log scale for the a_n axis.

Superpose a further graph of the alternative empirical function

$$a_n = 10^{0.245n-0.7}$$

how convincing do you find the Titius-Bode law?

4.49 Write down expressions for:

(a) The theoretical temperature T of a planet of albedo A at a distance D from the sun, the latter having radius $R_\odot$ and temperature $T_\odot$.

(b) The scale height H of the atmosphere of a planet of gravity g and temperature T for particles of mass m.

Hence show that the value of H for planets with the same values of A, m and g should vary with D according to $H \propto D^{-1/2}$.

One such planet at distance D_1 has $A_1 = 0$, while a second has $A_2 = 0.4$. At what distance D_2 must the second planet lie in order to have the same H as the first?

4.50 Explain how the earth is known to have a liquid core. If this core has radius 0.5 of the earth's radius, find the semi-angle ϕ of the zone over which no seismic S-waves are detected.

4.51 (a) Show that for planets of the same density, the surface gravity g varies with radius R as $g \propto R$.

(b) If gas giants have a density one quarter of the rocky terrestrials, show that a gas giant of 4 times the earth's radius would have the same g.

4.52 (a) Show that a planet of mass M comprised of constituent particles of mass m, formed by gravitational shrinkage to a radius R from a structure of much greater initial radius, would have a formation temperature T_0 given approximately by

$$T_0 \approx \frac{GMm}{k_B R}$$

if cooling processes are neglected. [6]

(b) Show also that such a young planet would then cool by radiation on a timescale τ given approximately by

$$\tau \approx \frac{k_B M}{4\pi\sigma R^2 m T_0^3} = \frac{k^4 R}{4\pi G^3 \sigma M^2 m^4}$$

if heating by the sun is relatively small in comparison. [8]

(c) Using $M = 4\pi R^3 \rho/3$, express T_0 and τ in the form

$$T_0 \approx \frac{4\pi G\rho R^2 m}{3k_B} \qquad \tau \approx \frac{9k_B^4}{64\sigma\pi^3 G^3 R^5 \rho^2 m^4}$$

[2,2]

(d) If all the terrestrial planets are taken to have the same m and ρ this would imply $T_0 \propto R^2$ and $\tau \propto R^{-5}$. Use the data on R in the handout to rank the inner planets and the moon in increasing order of: (a) T_0; and (b) τ in this approximation. [1,1]

4.53 The luminosity L (intrinsic brightness) of a comet varies with distance r from the sun according to $L \propto \frac{1}{r^a}$ where the constant $a > 2$ due to expansion of the comet as it approaches the sun. The flux F of light from the comet seen at distance d from the earth varies as the inverse square law $F \propto \frac{L}{d^2}$.

If F_0 is the flux from the comet when $d = r = 1$ AU show that the flux in general is $F = \frac{F_0}{d^2 r^a}$ when d, r are given in AU. Hence show that if the apparent magnitude of the comet when $d = r = 1\ AU$ is m_0, the magnitude in general is $m = m_0 + 5\log_{10} d + 2.5a \log_{10} r$.

A comet with $a = 6$, $m_0 = 1$ has an orbit with perihelion distance 0.3 AU. Calculate the brightest (lowest) apparent magnitude it can have when seen at perihelion from the earth.

NOTE - apparent magnitude m is related to flux F by $m = C - 2.5\log_{10} F$ where C is constant

4.54 A meteoroid has mass m and speed v on entering the atmosphere. If it stops in a distance L and all its kinetic energy goes into light show that its luminosity is about

$$L = \frac{mv^3}{2L}$$

and calculate L for $m = 0.01$ kg, $v = 30\text{kms}^{-1}$ taking $L = 10$ km.

4.55 If the earth had no atmosphere and a very dark surface ($A = 0$) its theoretical temperature would be 280 K.

To what value would A have to increase to result in this temperature falling to the freezing point 273 K of water?

If such an albedo change persisted and resulted in the whole surface of the earth becoming covered in snow ($A = 0.7$) to what temperature would the earth's surface fall?

Why is the earth warmer than these calculations suggest?

5 Stellar Motion

5.1 Define carefully the orbital elements of a planetary orbit. Explain briefly why they are not exactly constant, and describe in outline the changes that take place in the elements of the Moon's orbit about the Earth.

5.2 Show that the changes in a star's right ascension and declination *at transit* (in the sense observed minus true) are given by

$$\Delta\alpha = 0,$$
$$\Delta\delta = K \tan(\phi - \delta),$$

where K is the constant of refraction, which has the value of 60.4 arcsec under standard conditions of pressure and temperature.

5.3 Find the declination of the pole star after (a) one half and (b) one quarter of the precessional period. Assume that initially this star is located exactly at the north celestial pole and take the obliquity of the ecliptic $\epsilon = 23.5°$. {43°; 57°14′}

5.4 A star has ecliptic latitude and longitude $\beta = 60°$ and $\lambda = 75°$. Estimate the most favourable times of year for making stellar parallax measurements. What is the eccentricity of the parallactic ellipse described by this star?

5.5 An eclipsing binary has a constant apparent magnitude 4.35 between minima of its light curve. The apparent magnitude at the primary minimum is 6.28. Assuming that this minimum corresponds to a total eclipse, calculate the apparent magnitude of each component. What is the ratio of the two stars' brightnesses?

5.6 The pulsar PSR1913+16 has a variable apparent period. Analysis of this indicates that its radial velocity is periodically variable with an amplitude of $158\,\mathrm{km\,s^{-1}}$. The period of variation is 0.323 days. Treat this as a single-line spectroscopic binary and calculate the mass function.

The analysis of the pulses also reveals that the orbit precesses at 4.22° per year. This is interpreted as a general relativistic effect and indicates that the total mass of the binary system is 2.85 solar masses. Calculate a lower bound for the mass of the unobserved object.

5.7 Using the data given below calculate the length of the umbral cone of the Earth's and the Moon's shadow.

Radius of Sun	6.96×10^5 km
Radius of Earth	6.38×10^3 km
Radius of Moon	1.74×10^3 km
1 AU	1.50×10^8 km

5.8 The parallaxes of two stars are 0.074 and 0.047 arcsec. The stars have the same right ascensions, their declinations being 62° N and 56° N respectively. Calculate the distances of the stars from the Sun and the distance between them, in parsec. {8.0 pc;13.5 pc}

5.9 Find the masses of two stars in a binary system whose parallax is 0.075 arcsec and in which the apparent orbits of the stars are circular, with radii 4.5 and 1.5 arcsec relative to the centre of gravity of the system and are described with uniform motion with a period of 300 years. $\{1.42M_\odot; 4.27M_\odot\}$

5.10 A star X of the Taurus cluster, of declination $17°49'$N, has components of annual motion $\mu_\alpha = +0^s.0089$, $\mu_\delta = -0''.029$, and its angular distance from the convergent is $27°15'$. The radial velocity of a star Y of the cluster, at angular distance $26°57'$ from the convergent, is $+38.8$ kilometres per second. Calculate the distance, in parsecs, of the star X.

6 Observational methods

6.1e A galaxy is imaged by optical, infra-red, radio and X-ray telescopes. Describe briefly the likely differences in the features of the four maps. [5]

6.2e A comet displays a tail of 15° and it is planned to take photograph of it with a regular 35 mm camera with the picture format having dimensions 24 mm × 36 mm. Calculate the focal length of the lens that would be appropriate. [5]

6.3e Describe the physical reasoning behind the criterion that the minimum angle resolvable by a telescope may be represented by

$$\alpha = \frac{1.22\lambda}{D},$$

where λ is the wavelength of the radiation and D the diameter of the telescope. [5]

6.4e Define the term *parallactic angle*. Given that the smallest annual parallax that can be measured by ground based telescopes is 0.001 arcsec, calculate the maximum stellar distance that can be measured in parsecs and in metres. [5]

6.5e Explain why large objective telescope designs are no longer considered in preference to reflector systems. [5]

6.6e Describe the optical layout of a Cassegrain telescope. [6]

Explain the use of a Fabry or field lens in the design of a photoelectric photometer. [5]

Such an instrument is designed for use with a 2.2 m telescope with a focal ratio of $f/12$. Calculate the diameter in the focal plane of the field stop which limits the field of view to 15 arcsec. [6]

6.7e Explain why a dish type radio telescope provides poorer angular resolution than its optical counterpart. [7]

Describe the operation of a simple two-element radio interferometer. [5]

An interferometer with baseline of 1 km operates at a frequency of 3 000 MHz. Estimate the angular resolution of the system. [5]

6.8e Describe the basic photographic process for recording images of galactic clusters. [7]

Explain how magnitude values might be obtained by laboratory analysis of the plate. [7]

What is meant by the phrase: *the dynamic range of a photographic star plate is 5 magnitudes*? [4]

6.9e Describe a simple X-ray telescope. [6]

Describe the principle of a proportional counter for use as an X-ray detector. [6]

An X-ray source is at 50 pc and has a luminosity of 6×10^{31} W, chiefly liberated in the X-ray region. Estimate the detector pulse count production rate if the telescope has a collection area of 840 cm^2. [5]

(Assume X-ray photon energy $= 4\,\text{keV} \equiv 6.4 \times 10^{-16}$ J. $1\,\text{pc} = 3.086 \times 10^{16}$ m.)

6.10 (a) The CO molecule produces a spectral line at 230 GHz. What is the wavelength of the line? {1.3 mm}

(b) Using the identity $\lambda_{\text{nm}} = 1\,240/E_{\text{eV}}$, where λ_{nm} is the wavelength in nanometres and E_{eV} is the energy in electronvolts, calculate the energy of a photon associated with Hβ emission with wavelength of 486.1 nm. {2.55 eV}

(c) A band of X-ray radiation has a wavelength of 1 nm. What is the photon energy expressed in keV? {1.24 keV}

($1\,\text{GHz} = 10^9\,\text{Hz}$; $1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$)

6.11 A star provides $4 \times 10^{-17}\,\text{W m}^{-2}$ of Hα radiation at the bottom of the Earth's atmosphere. Calculate the number of Hα photons per second entering a telescope of 500 mm diameter.

Assuming that the main source of noise is the randomness associated with photon counting, estimate the signal-to-noise ratio of an Hα brightness measurement of the star using an integration time of 50 s.

6.12 (a) The 2.5 m Isaac Newton Telescope is used at $f/15$ with a 5 mm square CCD chip at its focus. Calculate the field of view that would be recorded within the picture format. [5]

(b) (i) An $f/10$ telescope with a focal length of 3 000 mm is used with an eyepiece of 20 mm focal length. What is the magnifying power of the system? Would this magnification allow all the collected light to enter the pupil of the eye? [6]

(ii) A double star has a separation of $3''$ (3 arcsec) on the sky. What would be the physical separation of the images in the telescope focal plane? [4]

(iii) The limiting magnitude for this telescope is 13.6. Based on this figure, what limiting magnitude might be expected for a telescope of 500 mm diameter? [5]

6.13 (a) A 3-mirror coudé telescope feeds an all-reflection spectrometer comprising a collimator, a diffraction grating and a camera. The entrance slit of the instrument allows through only 50 % of the light contained in star images. Assuming the reflectivity of all the mirrors to be 80 % and the efficiency of the reflection grating to be 50 % , estimate the overall optical transmittance of the system. {8.2%} [4]

(b) An astronaut observes a star while in a launch vehicle and records its magnitude as 4.5 when it is at a zenith distance of 25°. Given that the zenith extinction is 0.5 mag, what magnitude would the star appear to have when viewed from the module in orbit? {3.95} [4]

(Bouger's law for atmospheric extinction may be written as $m(\zeta) = m_0 + \Delta m \sec \zeta$.)

6.14 A 3-mirror coudé, telescope has a primary mirror of 2 m diameter and is used in the infra-red, the pass band of the detector having a wavelength of 1.2 μm. What is the theoretical angular resolution of the telescope?

6.15 Explain why the sizes of in-focus images of stars photographed by a telescope reflect their flux densities. [6]

From a calibration study using standard stars in the field, it is found that a measured image diameter, d (in μm), can be related to a magnitude by

$$m = 10 - \log_{10} d.$$

Two stars are measured to have image diameters of 100 and 150 μm respectively. What is their magnitude difference? {0.176 mag.} [4]

6.16 For a photographic plate with a gamma of $\gamma = 0.6$ it is found that reasonable exposures cover a density range of 2. If a star of 8$^{\text{th}}$ magnitude is at the limit of over exposure, calculate the brightness of the faintest star that can be recorded satisfactorily so allowing a good measurement of its brightness.

6.17 Compare and contrast the use of photographic plates and photomultiplier tubes as a means of performing stellar observations.

It is planned to observe photometrically 25 stars within a stellar cluster to an accuracy of ± 0.05 magnitudes. Estimate whether it is better in terms of efficiency of telescope time to perform the measurements using photography or whether a photoelectric photometer might be used.

6.18 Sketch out the design of an imaging X-ray telescope.

Describe the principle of a Proportional Counter as used in X-ray Astronomy.

An X-ray collimating telescope comprises a honeycomb of tubes with a total collecting area of $0.5\,\mathrm{m}^2$. A source has an emission line close to 10 nm and provides a flux of $10^{-18}\,\mathrm{W\,m^{-2}}$ at this wavelength. Estimate the count rate from a proportional counter when this telescope system is directed to the source.

6.19 The Lovell radio dish at Jodrell Bank has a diameter 80 m and the receivers can be tuned to 1 420.4 MHz, with a band-pass of 10 MHz. A point radio source of 2.5 janskys (Jy) is observed. What is the theoretical angular resolution of this system? What is the power available for detection?

$(1\,\mathrm{Jy} = 10^{-26}\,\mathrm{W\,m^{-2}\,Hz^{-1}})$

6.20 Describe the principle of a microchannel plate as might be used in deep ultraviolet astronomy. Also describe the principle of a proportional counter as used in X-ray astronomy.

An X-ray collimating telescope comprises a honeycomb of tubes with a total collecting area of $0.5\,\mathrm{m}^2$. A source has an emission line close to 10 nm and provides a flux of $10^{-18}\,\mathrm{W\,m^{-2}}$ at this wavelength. Estimate the count rate from a proportional counter when used with the telescope which is directed to the source.

6.21$_e$ The Hα line (656 nm) of a star is observed in emission and the radiation from a narrow bandpass centered on the line provide a flux of $10^{-17}\,\mathrm{Wm^{-2}}$ at the bottom of the Earth's atmosphere. A monochromator tuned to this wavelength is attached to a 2.5 m telescope. Calculate the photon collection rate of the system assuming that there are no losses in the instrumentation. [5]

6.22$_e$ A CCD chip comprises an array of $2\,048 \times 2\,048$ elements each of 15 μm square. It is to be used on a telescope of 10m focal length. What is the width of the field of view (arcsec) that the system would capture? [5]

[1 radian = 206 265 arcsec]

6.23e Sketch out the optical layout of a grating stellar spectrometer indicating the important parts of the system. [7]

A spectrometer with spectral resolving power of 5×10^5 is used to determine the radial velocities of stars. What is the typical minimum stellar velocity that could be detected with the system? [4]

What is meant by the RLD (reciprocal linear dispersion) of a spectrometer? [2]

A CCD with 300 pixels per row, each element being 15 μm wide, is attached to a spectrometer with an RLD of 25 Å mm^{-1}. What is the spectral range covered in a single exposure? [4]

6.24e Describe the configuration of an X-ray telescope suitable for imaging. [5]

Such a system with a collection aperture of 30 cm diameter is designed to take X-ray images at energies of 2 keV. Using the Rayleigh criterion, estimate the angular resolution of the instrument in arcsec. [7]

Describe the principle of a detector system suitable for operation with the above instrument. [5]

[1 radian= 206 265 arcsec.]

6.25e Two stars radiate as black bodies with maximum brightness at wavelengths, λ_{max}, of 500 nm and 700 nm. It is also known that the first star has a radius three times larger than the second. What is the ratio of their luminosities? [5]

6.26 A star provides flux of $6 \times 10^{-17}\,\mathrm{W\,m^{-2}}$ of optical radiation in a wavelength passband centered on 500nm at the bottom of the Earth's atmosphere. A 2.5 metre telescope is used to perform photometry.

(a) Determine the theoretical angular resolution of the telescope.

(b) Calculate the flux entering the telescope.

(c) Estimate the number of photons per second collected by the aperture.

(d) Given that the transmittance of the optical system is 60% and the quantum efficiency of the photo-electric detector is 15%, determine the signal in terms of recorded photo-electrons per second.

(e) Calculate the observing time required to obtain a signal-to-noise ratio of a measurement of 10.

(f) Prior to the display of the analogue signal, the electronic system provides a gain of 10^6. Calculate the current flow to be converted as the final display.

6.27 Sketch out the optical elements and their layout for a basic spectrometer. [9]

Estimate the typical transmission efficiency of an optical spectrometer. [3]

Define the term *Spectral Resolving Power*. [3]

A spectrometer has a 25 mm diffraction grating with a ruling of 1 800 lines per mm and is used in first order. Estimate the smallest stellar radial velocity that might be detected with this instrument. [5]

6.28 A 4 m Cassegrain telescope works at $f/15$.

(a) Determine the theoretical resolving power of the telescope. [3]

(b) What is the size of the recordable field by a 1 cm$\times$1 cm square CCD chip in the telescopes focal plane? [4]

(c) Given that the quantum efficiency of the CCD is 60%, estimate the overall telescope/detector efficiency. [4]

(d) If the telescope were to be used visually, estimate the limiting magnitude. [3]

6.29 A 50 cm space telescope in near Earth orbit is equipped with a filter, centred at 200 nm, with a passband which is 2 nm wide. The combined transmission efficiency of telescope and filter is 1%. It is used to observe a star at a distance of 15pc. At a wavelength of 200nm, the star's specific luminosity (luminosity per unit wavelength) is approximately 7×10^{22} Watts nm^{-1}. How many photons per second from the star will be transmitted by the telescope and filter? [8]

6.30 A Newtonian telescope has a primary mirror of diameter 12 cm and a secondary flat. Each of these has a reflection efficiency of 0.70. Attached to the telescope is an eyepiece with transmission efficiency of 30%. The telescope is pointed at a star which gives a flux of visible light at Earth of 10^{-14}W m^{-2}.

(a) what is the combined efficiency of this system?

(b) estimate how many photons per second will be (i) collected by the telescope primary and (ii) transmitted by the telescope and eyepiece combined.

Clearly state any assumptions that you make [6]

6.31 The globular cluster M13 is at a distance of 7.20 kpc, and has a measured angular size at Earth of 16.6 arcseconds. The energy flux received from M13 at the top of the Earth's atmosphere is 2.45×10^{-10} W m^{-2}.

(a) What is the diameter of M13 in light years?

(b) Assuming that M13 is spherical, what is the *solid angle* it subtends at Earth?

(c) Assuming that M13 is made up of stars all of approximately solar luminosity, calculate how many stars M13 contains.

[10]

6.32 To make a reliable measurement of a faint stellar spectral line, it is necessary to have a detected signal-to-noise ratio (SNR) of 10. Using a particular telescope and detector combination, and an integration time of 10s leads to a SNR of 5. How long an integration time is necessary to achieve the required SNR value?

Updates to the optics in the telescope result in an improved throughput, such that the total transmission efficiency increases by a factor 2. With this new system, how long an integration time, for the same star, will give the required SNR of 10? [6]

6.33$_e$ A spectrometer has a reciprocal linear dispersion (RLD) of 20 Å mm^{-1}. It is used to record the spectrum of a star with an estimated recessional velocity of 50 km s^{-1}. Calculate the displacement of the recorded lines relative to a laboratory reference spectrum in the wavelength region around 5000 Å. (June 1999 degree exam) [5]

6.34e (a) Describe the principal components of a photo-electric photometer (optics and electronics). [6]

(b) A star provides a flux of 10^{-16}W m^{-2} per Å bandpass in the spectral region around 500 nm and is observed with a telescope of 1m diameter. The telescope has a transmission efficiency of 60% and the detector a quantum efficiency of 25%. A filter limits the bandpass to 10 Å. Estimate the photoelectron count rate from the system [7]

(c) Assuming that the limiting noise on photometric records is photon shot noise, calculate the integration time necessary to obtain measurements which are reproducible to $\pm$ 1%. [4]

(June 1999 Degree Exam)

7 Stellar Astrophysics

7.1e What are the main differences between population I stars and population II stars? Why should we expect most of the population I stars to be in the galactic plane? [5]

7.2e Write down Rydberg's equation, giving the wavelengths of spectral lines from atomic hydrogen. Calculate the wavelengths of the first three spectral lines in the hydrogen Balmer series (i.e., Hα, Hβ and Hγ), given the Rydberg constant, $R_\infty, = 1.097 \times 10^7\,\mathrm{m}^{-1}$. [5]

7.3e Draw a sketch of the general Hertzsprung-Russell diagram for stars. Label the axes suitably and clearly mark those regions corresponding to main sequence stars, red giants and white dwarfs. [5]

7.4e Briefly, what are:

(a) Cepheid variables

(b) T Tauri stars

(c) eclipsing binary stars

(d) planetary nebulae? [5]

7.5e If we assume that a star radiates as a blackbody, what is the relationship between its luminosity, radius and temperature? [5]

7.6e Explain how measurements of a star's colour index and spectral type can each be used to determine its effective temperature. [5]

7.7e Write brief notes on any three of the following topics, highlighting their context within stellar astrophysics:

(a) The determination of stellar mass

(b) The determination of the effective surface temperature of stars

(c) Line production and the spectral classification of stars

(d) Stellar evolution off the main sequence
(e) Variable stars. [17]

7.8e The relationship between the mass, M, and luminosity, L, of a lower-main sequence star is roughly $L \propto M^{3.5}$. If a star begins to leave the main sequence when 20 % of its hydrogen has been converted into helium, estimate the time spent on the main sequence by a star of $10M_{\odot}$ and one of $0.1M_{\odot}$, given that the proton-proton reaction converts 0.7 % of the mass of the hydrogen fuel actually used into energy. You may assume that the star is initially 100 % hydrogen. [10]

Discuss the significance of your result and use it to describe how the Hertzsprung-Russell diagram for a cluster of stars (all of the same age, but differing in mass) evolves over time. [7]

7.9 (a) Given that the Sun-Earth distance is 1.5×10^8 km, the angular diameter of the Sun is 32 arcmin, and the solar constant is 1.36×10^3 W m^{-2}, determine the Sun's luminosity, $L_{\odot}$, and effective surface temperature T_e. {3.85×10^{26} W; 5 771 K} [7]

(b) The luminosity of a star is calculated, from its apparent magnitude and distance, to be $16L_{\odot}$. Its surface temperature is estimated to be 5 000 K. Estimate the radius of the star. [6]

(c) A $0.8M_{\odot}$ white dwarf star has an effective temperature of 5 000 K and luminosity of 10^{24} W. Calculate the radius and mean density of the star. {4.7×10^4 km; 3.68×10^6 kg m^{-3}} [7]

7.10 An eclipsing binary has a constant apparent magnitude 4.35 between minima and apparent magnitude 6.82 at primary minimum. Assuming that the eclipse is total at primary minimum, calculate the apparent magnitudes and the relative fluxes received from the two components. (Hint: remember that the flux,F, received from a star is related to its apparent magnitude, m, by $m = -2.5 \log_{10}(F/F_0)$, where F_0 is the flux from a zero magnitude star.) {0.116}

7.11 Two stars are seen to be in a circular binary system that is eclipsing, indicating that we are seeing them almost in the plane of their orbit. An astronomer observes the Doppler shifts of spectral lines from the stars and deduces the following radial (i.e., line-of-sight) velocities for the stars as a function of time:

Time (years)	v_r of star 1 (km s^{-1})	v_r of star 2 (km s^{-1})
0.0000	40	40
0.0005	130	15
0.0010	160	0
0.0015	130	15
0.0020	40	40
0.0025	−50	65
0.0030	−80	80
0.0035	−50	65
0.0040	40	40
0.0045	130	15
0.0050	160	0
0.0055	130	15
0.0060	40	40

(a) On one graph, plot the velocity curves of the two stars as a function of time. Hence determine the period of the system, P, and from the mean velocities of the stars the recession velocity of the whole system, v_0. {1.461 days; 40 km s^{-1}} [5]

(b) Show that the separation of two stars in such a system can be written as $R = P(v_1 + v_2)/(2\pi)$, where v_1 and v_2 are the orbital speeds of the stars, and determine this separation. (Remember to take account of v_0.) {0.02 AU} [5]

(c) Show also that the separation of the stars is related to the sum of their masses, m_1 and m_2 by $R^3/P^2 = G(m_1 + m_2)/(4\pi^2)$, where G is the gravitational constant. Hence determine the total mass of the system. [5]

(d) Finally, show that the ratio of the masses equals the reciprocal ratio of their orbital speeds. From this and the result of part (c) above, calculate the individual masses of the two stars. [5]

7.12e What is the Stefan-Boltzmann equation, relating the power emitted from a blackbody per unit surface area to its temperature, T? Explain how the relationship can be used to relate the temperature, radius and luminosity of a star. [6]

Explain how an eclipsing binary system can be identified from its characteristic light curve, distinguishing between the *primary* and *secondary* eclipse. [6]

An eclipsing binary system comprises two stars of temperatures $T_1 > T_2$ and radii $R_1 < R_2$. Assuming that the stars can be modelled as uniformly bright flat discs (area πR^2) in the plane of the sky, show that the temperatures of the stars are related by

$$\frac{F - F_\mathrm{p}}{F - F_\mathrm{s}} = \frac{T_1^4}{T_2^4},$$

where F_p and F_s are the fluxes received from the whole system during the primary and secondary eclipses, and F is the flux received when there is no eclipse. [5]

7.13 (a) Distinguish between the spectrum of a typical O star and a typical red giant. What information about the nature of these stars would you be able to deduce from a study of their spectra? [7]

(b) Helium shows an absorption line at a wavelength of 447.1 nm due to the excitation of an electron in the atom already at an energy level of −3.5 eV (i.e., 3.5 eV below the ionisation level). Calculate the energy of the electron after absorption. {−0.73 eV} [7]

(c) The ionisation potential of He is 24.47 eV. At what temperature would you expect to observe the appearance of singly ionised helium (He II) lines, and why? {15 800 K} [6]

7.14 A star has a mass of $6M_\odot$.

(a) Show, by a sketch, how energy is transported in the interior of this star during its main sequence phase. [3]

(b) Describe briefly the nuclear reaction that powers a star of this mass. Why do stars of a much lower mass fuse hydrogen via a different chain reaction? [4]

(c) Describe in outline the formation of such a star from a dense cloud. Your description should finish when the star joins the main sequence. How will its luminosity vary during this process? [10]

(d) The reactions of the CNO cycle are:

$$\begin{aligned} {}^{12}\mathrm{C} + \mathrm{p}^+ &\rightarrow {}^{13}\mathrm{N} + \gamma \\ {}^{13}\mathrm{N} &\rightarrow {}^{13}\mathrm{C} + \mathrm{e}^+ + \nu \\ {}^{13}\mathrm{C} + \mathrm{p}^+ &\rightarrow {}^{14}\mathrm{N} + \gamma \\ {}^{14}\mathrm{N} + \mathrm{p}^+ &\rightarrow {}^{15}\mathrm{O} + \gamma \\ {}^{15}\mathrm{O} &\rightarrow {}^{15}\mathrm{N} + \mathrm{e}^+ + \nu \\ {}^{15}\mathrm{N} + \mathrm{p}^+ &\rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He} \end{aligned}$$

Show that the overall reaction is

$$4\mathrm{p}^+ \rightarrow {}^{4}\mathrm{He} + 2\mathrm{e}^+ + 3\gamma + 2\nu,$$

and that ${}^{12}\mathrm{C}$ acts as a catalyst. [3]

7.15 One hundred years ago astronomers accepted the idea that the origin of the Sun's luminosity was gravitational contraction. Assume that the gravitational potential energy of the Sun is given by

$$\Omega = -\frac{\alpha G M_\odot^2}{R_\odot},$$

where α is a numerical constant. Hence, using the virial theorem, show that the luminosity would be given by

$$L_\odot = \frac{\Omega}{2R_\odot}\frac{\mathrm{d}R_\odot}{\mathrm{d}t}.$$

Taking $\alpha = 1$, and the given values for $L_\odot$, $M_\odot$ and $R_\odot$, work out the change in radius over 1 year.

7.16 Describe the main features of the Hertzsprung-Russell diagram, and show on your diagram the position of

(a) Main Sequence stars

(b) Red Giant stars

(c) White Dwarf stars.

White dwarf stars have absolute magnitude of between 6 and 8. What will be the range of apparent magnitude of such stars in a globular cluster at a distance of 10 kpc?

7.17 Many astrophysical objects (e.g., solar and stellar flares) attain such high temperatures that even heavy elements like iron (atomic number 26) are almost completely ionised. Using the Bohr theory, calculate the wavelengths of the first four spectral lines and the series limit for electrons undergoing transitions to the ground state of 25 times ionised iron (i.e., iron which has lost 25 of its 26 electrons). What part of the electromagnetic spectrum are these lines in?

The series limit of a set of lines from 25 times ionised iron occurs at 2.16 nm. To which level are the transitions in this series taking place?

7.18 A spherical star of mass M and radius R is in hydrostatic equilibrium.

- If the density is given by
$$\rho = \rho_c\left(1 - \frac{r}{R}\right),$$
where ρ_c is the central density, show (by determining the mass, M, of the star) that
$$\rho_c = \frac{3M}{\pi R^3}.$$
- Use the equation of hydrostatic equilibrium to find the central pressure, p_c, of the star.
- Assuming the star is made entirely of hydrogen, use the ideal gas law to determine the central temperature of this star, T_c, and evaluate T_c for a star with the same mass and radius as the Sun. (hint: remember electrons are particles too!)
- Why is Jupiter not a star? ($M_{\rm Jupiter} = 0.001M_\odot$, $R_{\rm Jupiter} = 0.1R_\odot$)
- Would you expect to get a higher or lower central temperature if you considered a more realistic chemical composition for the star?

7.19 The brightest main sequence star of a certain cluster has absolute magnitude 0, and a mass of $4M_\odot$. Given that a star leaves the main sequence when its helium core constitutes $1/5$ of the total mass of the star, initially of pure hydrogen, calculate the age of the cluster from the data given below.

Sun's luminosity	4×10^{26} W
Sun's absolute magnitude	$+4.6$
Sun's mass	2×10^{30} kg
Energy released per unit mass H→He	6×10^{14} J kg^{-1}

7.20 Taking the mass luminosity (M, L) relationship for main sequence stars to be $L/L_\odot = (M/M_\odot)^{3.5}$ and assuming that a star leaves the main sequence after burning 20 % of its hydrogen, show that the main sequence lifetime of the Sun is 1.93 times longer than that of a star of mass equal to the limiting mass of a white dwarf star. (Take this limit to be $1.3M_\odot$). [4]

7.21 Calculate the fraction of the mass of ^{4}He that can be released through the nucleosynthesis of ^{12}C through the triple α reaction
$$3\,{}^4\text{He} \rightarrow {}^{12}\text{C},$$
given that the mass of ^{4}He is 4.003 amu and the mass of ^{12}C is 12 amu. [2]

The core of a $2M_\odot$ star consists of pure ^{4}He and constitutes $1/5$ of the total mass of the star. Assuming the star has a luminosity $L = 10L_\odot$, estimate the maximum continued lifetime of the star. [4]

7.22 A star 5 times the mass of the Sun explodes as a supernova at a distance of 2 000 parsec. It is postulated that in the rapid progression of the conversion of hydrogen to iron, 80 % of the hydrogen is first converted to helium via the proton-proton cycle mechanism. Estimate the number of neutrinos that would pass through your body as a result of this initial phase of the explosion. {1.4×10^{17} m^{-2}}

Assuming the explosive event takes just a few seconds, how does this rate of arrival compare with the regular dosage provided by the Sun (see your notes for this figure)? [10]

(Assume that the original star comprises 100 % hydrogen.)

7.23 Assuming that stars radiate as blackbodies, show how the Stefan-Boltzmann law relates the radius, luminosity and effective temperature of a star. [5]

Stars towards the middle of the main sequence on the Hertzsprung-Russel diagram approximately obey the *mass-luminosity relation*, $L \propto M^3$, where L is a star's luminosity and M its mass. With the simplifying assumption that these stars all have approximately the same mean densities, show that their luminosities and temperatures would be related by

$$\log L = \frac{36}{7} \log T + \text{constant} \quad ,$$

i.e., that the main sequence would have a slope of about -5.1 on the Hertzsprung-Russell diagram. [10]

(Note that in reality, the mean densities of low-mass stars are significantly greater than the mean densities of high-mass stars.)

Draw a schematic H-R diagram, labelling the axes in terms of $\log(L/L_\odot)$ and $\log(T/T_\odot)$, and include a line for the main sequence with the slope you have just derived. Show also lines of constant stellar radius. [5]

7.24 Use the lecture handout giving data on stellar spectral types to answer the following:

Rigel A is the prominent bluish-white star in the constellation of Orion. Its spectrum is such that it is classified as a B8 star.

(a) Estimate its effective temperature.

(b) How would you expect the relative strengths of the Hydrogen Balmer, He I and Ca II lines to compare in this star? (This is one of the criteria by which spectral type is identified.)

(c) Spectroscopic studies lead to an estimate of $1.4 \times 10^5\ L_\odot$ for its luminosity. How would you expect its spectral lines to differ from those of a far less luminous star of the same surface temperature?

(d) Calculate the radius of Rigel A in solar radii.

7.25e Assuming that all stars obey the Stefan-Boltzmann law, show how lines of constant stellar radius appear on the Hertzsprung-Russell diagram. Explain how red giants and white dwarfs fit into this diagram. [5]

7.26e A spectroscopic double star emits a spectral line at 441 nm that splits into two components of maximum separation 0.2 nm every 10.25 days. What is the minimum spatial separation of the two stars, assuming they are in circular orbits? [5]

7.27e Explain why the Hα absorption line (the first line in the Balmer series) is weak for both very hot and very cool stars. Which stellar spectral type has the strongest hydrogen absorption lines? [5]

7.28e Stars on the main sequence have masses of between about 0.1 and 50 $M_\odot$.

(a) Show where the least and most massive of these stars are found on the Hertzsprung-Russell diagram. [4]

(b) What factors influence the cut-offs at the two extremes? [3]

(c) How does the position of a star change on the main sequence as it ages? [2]

(d) Approximately what range in luminosity would youu expect for these stars? [4]

(e) If the Sun has an expected lifetime of about 10^{10} years, estimate the lifetimes of stars at the extremes of the main sequence. [4]

8 Compact objects

8.1e Draw a $(\log P, \log r)$ diagram for cold dense matter, labelled with the main sources of pressure involved in each part. Show the tracks of contracting stellar remnants and indicate the locations of white dwarf stars, neutron stars, and black holes. [5]

8.2e Explain briefly why a star can never reach a final static equilibrium state unless it develops an internal pressure which does not depend on temperature. Name the two classes of such cold stellar remnants, identify the range of masses of each. What do stellar remnants heavier than these become? [5]

8.3e What observational features characterise pulsars? Derive an expression for the minimum rotation period of a mass M, radius R and express this in terms of density ρ. [5]

8.4e What is meant by a compact accretion X-ray source ? Write down an expression for the luminosity, $L_{\rm acc}$, due to accretion and derive an expression for the limit set by radiation pressure. [5]

8.5e Explain briefly why pulsars pulse. The period of the Crab pulsar is 33 ms. If the number of pulses received on successive days drops by 3 per day calculate its slowdown rate, $\dot{P}$, in seconds per day. [5]

8.6e Explain why pulsars must be small massive objects and why white dwarf stars are ruled out in favour of neutron stars. [5]

Show how neutron stars acquire such high rotation rates and strong magnetic fields and obtain an expression for the minimum rotation period. [6]

Due to internal restructuring the radius R of a neutron star is reduced to $0.9R$. Assuming mass, angular momentum and total magnetic flux are conserved, obtain the factors by which the following change:

(a) rotation rate [2]

(b) magnetic field, B. [2]

8.7e Given the expressions $p = \hbar/\Delta x$, $P = \beta n E$ and $E_{\rm e} = p^2/(2m_{\rm e})$, where the symbols have their usual meanings, derive the quantum electron pressure in a hydrogen gas at density ρ:

$$P_{\rm Qe} = \frac{\beta\hbar^2}{2m_{\rm e}}\left(\frac{\rho}{m_{\rm p}}\right)^{5/3}.$$

[6]

Given also that the central pressure needed to balance gravity for a spherical mass M is

$$P_c = \left(\frac{\pi}{6}\right)^{1/3} G M^{2/3} \rho^{4/3},$$

derive the density of a white dwarf star:

$$\rho_{wd} = \frac{4\pi}{3} \frac{G m_p^5 m_e^3}{\beta \hbar^6} M^2.$$

[5]

Write down the corresponding expression for a neutron star and state the source of pressure. [3]

Hence show that $R_{ns}/R_{wd} \simeq m_e/m_p$. [3]

8.8 Two degenerate white dwarf stars have masses $\mathcal{M}_1$, $\mathcal{M}_2$ and surface temperatures T_1, T_2. Assuming them to radiate like blackbodies (i.e., bolometric luminosity $L_{bol} = 4\pi R^2 \sigma T^4$), show that their absolute bolometric magnitudes M_1, M_2 satisfy

$$M_1 - M_2 = \frac{5}{3} \log_{10} \frac{\mathcal{M}_1}{\mathcal{M}_2} - 10 \log_{10} \frac{T_1}{T_2}.$$

8.9e Briefly describe the observational evidence for the existence of white dwarfs and neutron stars. [6]

By estimating the gravitational attraction between two halves of a uniform sphere, show that the pressure within a star of mass M and radius R is approximately

$$P \simeq \frac{GM^2/(4R^2)}{\pi R^2},$$

and hence that $P \propto M^{2/3}\rho^{4/3}$, where ρ is the stellar mass density. [4]

In white dwarfs, this pressure is balanced by an outward pressure that increases as $\rho^{5/3}$. Describe the source of this pressure and use the relation to show that, in equilibrium, the radius of white dwarfs varies as $R \propto M^{-1/3}$. [4]

Compare this with the corresponding result for stars of constant density. [3]

8.10 Sirius B is a white dwarf companion to the brightest star in the night sky, Sirius A. It has an effective temperature of 29 000 K and a mass of 1 $M_\odot$.

Use your lecture notes and the physical constants at the front of the problems handbook to estimate:

(a) the radius of Sirius B, R {4.8×10^6 m}

(b) its mean density, ρ {4.3×10^9 kg m^{-3}}

(c) its luminosity, L. Express this as a multiple of the solar luminosity, $L_\odot$, and comment on the value {$0.03\,L_\odot$}

(d) The electron degeneracy pressure at its centre, P_Q, expressed in pascals (1 Pa = 1 N m^{-2}). {1.96×10^{22} Pa}

The thermal gas pressure that supports main-sequence stars against gravitational collapse will also contribute slightly to supporting white dwarfs. Taking the thermal pressure at the centre of a white dwarf to be $P_T = 2kT\rho/m_p$, where T is the temperature k is Boltzmann's constant and m_p the mass of a proton, estimate its value and compare it to the electron degeneracy pressure calculated above. {2×10^{18} Pa}

At what temperature would these pressures be equal? {2.75×10^8 K}

Assuming Sirius B consists of a soup of electrons and protons, what is the mean separation of the protons? How does this compare with the separation of atoms in a 'normal' solid? {$\sim 7 \times 10^{-13}$ m}

[$\hbar = 1.054 \times 10^{-34}$ J s. Note that any answer close to those quoted is acceptable. In fact the inside of a white dwarf is thought to be considerable hotter than the surface.]

8.11 Show that the density of a neutron star of M just above M_{Ch} is about $(m_p/m_e)^3$ times larger than that of a white dwarf of M just below M_{Ch}. Hence show that the minimum rotation period of a white dwarf is $(m_p/m_e)^{3/2}$ times larger than that of a neutron star of comparable mass. [12]

The angular velocity ω of rotation of a self-gravitating sphere of mass M, radius R, is limited by centrifugal break-up to $\omega < \omega_1$ as described in your notes. In addition, relativity forbids any part of the rotating body to move faster than the speed of light, c, so setting another limit $\omega < \omega_2$. Show that the centrifugal limit, ω_1, always applies first (i.e., show that $\omega_1 < \omega_2$) since, for the reverse to be true would imply $R < R_S/2$, where R_S is the Schwarzschild radius, – i.e., the body would be a black hole. [8]

8.12 The period of the Crab pulsar is 33 ms and the luminosity associated with it is 2×10^{31} W. Assuming the neutron star to have about one solar mass and a radius of 5 km, find the time scale on which the Crab pulse period is increasing. (l year = 3.1×10^7 s)

8.13 A binary star system comprises an ordinary star which shows periodic Doppler shifts of amplitude $\Delta\lambda = 0.168$ nm in an optical line of $\lambda_0 = 600$ nm, and a pulsar the apparent period of which varies with an amplitude $\Delta t = 1.26$ ms about a mean of $t_0 = 0.9$ s. If the primary star has a spectral type indicating a mass of $10M_\odot$, find the mass of the companion in units of $M_\odot$.

Given also that the orbital period is 2.78 hours, find the inclination of the system.

8.14 Show that if matter accretes *directly* at a rate $\dot{M}$ on to a spherical mass and if its energy is radiated off as blackbody radiation, the temperature resulting is

$$T = \left(\frac{G\dot{M}\rho}{3\sigma}\right)^{1/4},$$

where σ is the Stefan-Boltzmann constant and ρ the mean density of the object. Calculate T for a white dwarf star of $1M_\odot$ and for a neutron star of $2M_\odot$ using the relevant values or expressions for G, σ and ρ from your notes. Assume mass increases by accretion on a timescale of 10^8 years.

8.15 Show that the mass M of a spherical mass accreting matter at the Eddington limit is increasing on a timescale

$$\tau = \frac{\sigma_p}{4\pi m_p c}\frac{M}{R},$$

where σ_p is the photo absorption cross-section for the accreting protons.

8.16 It was at one time proposed that the Sun was kept luminous by the potential energy released due to infall ('accretion') of meteors. Given that the mass of the Sun is 2×10^{30} kg, the radius of the Sun is 7×10^8 m, and the luminosity 4×10^{26} W, calculate approximately how long it would take the mass of the Sun to double due to this process. (l year $= 3 \times 10^7$ s).

Can you suggest at least two observations which would refute this idea?

8.17 In your notes, degenerate objects like white dwarfs and neutron stars have been described in terms of 'cold' degenerate pressure, P_{Qe}. In practice, such stars are quite hot when they form but at high densities, the thermal gas pressure, $P = \rho kT/(\mu m_p)$, is assumed to be negligible. Taking the expression for the degenerate electron pressure P_{Qe} etc. from your notes, show that the ratio of thermal to degenerate pressure at temperature T for a white dwarf of mass M is

$$\frac{P}{P_{Qe}} = \frac{2\beta}{\mu}\left(\frac{3}{4\pi}\right)^{2/3}\frac{\hbar^2 kT}{G^2 m_e m_p^{8/3}}\frac{1}{M^{4/3}}.$$

Using the values of the constants from your notes, calculate what temperature T a white dwarf of $M = M_\odot$ would have to have before P was roughly equal to P_{Qe}. {3.2×10^9 K}

(For this purpose take $\mu = \beta = 1$.)

8.18 Taking the gravitational energy of a uniformly dense sphere of mass M, radius R to be $\Omega = -3GM^2/(5R)$, show using the expression for R_{ns} from your notes that the gravitational energy of a neutron star of mass M is

$$\Omega = \frac{-3}{5\beta}\left(\frac{4\pi}{3}\right)^{2/3}\frac{G^2 M^{7/3} m_p^{8/3}}{\hbar^2},$$

and calculate Ω for a neutron star of $M = 2M_\odot$ (taking $\beta = 1$). Compare this with the Sun's output $L_\odot$ over 10^{10} years.

A planet of the same mass m (2×10^{27} kg) and radius r (7.2×10^7 m) as Jupiter orbits a giant star at a distance $D = 8.2 \times 10^{11}$ m. This star turns supernova and releases energy $|\Omega|$ by the collapse of a $2M_\odot$ core into a neutron star. If 1 % of $|\Omega|$ goes into radiation, calculate the planet's rise in temperature. [To do this, assume that the planet intercepts a fraction $\pi r^2/(4\pi D^2) = (r/2D)^2$ of the total radiation and that the rise in temperature ΔT is given by $(m/m_p)kT = U$ where U is the total energy absorbed, $k = 1.4 \times 10^{-23}$ J K^{-1} is Boltzmann's constant, and m_p is the proton mass (so that m/m_p is the number of 'atoms' in the planet)].

What would be the effect on the planet?

The other 99 % of energy goes into neutrinos. If each of these has energy 1 MeV, calculate the total number of neutrinos which would pass through you (assume an area of 1 m^2) if the explosion took place in the Larger Magellanic Cloud (distance 50 kpc).

8.19 The angular velocity of rotation, Ω, of a self-gravitating sphere of mass M, radius R, is limited by centrifugal break-up to $\Omega = \Omega_1$ as described in your notes. In addition, relativity forbids any part of the rotating body to move faster than the speed of light c, so setting another limit $\Omega = \Omega_2$. Show that the centrifugal limit, Ω_1, always applies first (i.e., show that $\Omega_1 < \Omega_2$) since, for the reverse to be true would imply $R < R_S/2$, where R_S is the Schwarzschild radius, so that the body would be a black hole.

8.20 A star of mass M and radius R, made of ionised hydrogen, collapses without mass loss to become a white dwarf of radius $R_{wd} \ll R$, and in doing so loses gravitational energy Ω. Taking the gravitational energy at radius R to be $\Omega = -3GM^2/(5R)$, and assuming that half of the energy lost goes into heat, show that the star's temperature increases during collapse by an amount

$$T = \frac{GMm_p}{10kR_{wd}},$$

where k is Boltzmann's constant, G the constant of gravitation, and m_p the proton mass.

Hence calculate the mean temperature at formation of a white dwarf of 1 solar mass, and radius 2.3×10^6 m, neglecting the initial temperature. If the surface of the star had this temperature in what wavelength range would you expect it to emit?

8.21 (a) Each of two white dwarf stars has mass M_1 and radius R_1 and is rotating at angular velocity ω_1. If these stars coalesce into a single white dwarf of mass $2M_1$, and corresponding white dwarf radius, and if rotational angular momentum ($\sim MR^2\omega$) is conserved, show that the angular velocity of the new single star is

$$\omega_2 = \omega_1(M_2/M_1)^{2/3} = 2^{2/3}\omega_1.$$

If the initial stars are at their critical centrifugal break-up angular speed of

$$\omega_1 = (GM_1/R_1^3)^{1/2},$$

show that the new star rotates below its critical break- up ω value by a factor of $2^{1/3}$.

(b) A rotating neutron star gradually increases in mass due to accretion of matter. Angular momentum is conserved in the process and the accreted matter initially has no angular momentum. Use the $R(M)$ relation for neutron stars to show that the angular speed ω, and rotational kinetic energy E, vary with M according to

$$\frac{\omega}{\omega_0} = \frac{E}{E_0} = \left(\frac{M_0}{M}\right)^{1/3}$$

while the (negative) gravitational binding energy Ω varies as

$$\frac{\Omega}{\Omega_0} = \left(\frac{M}{M_0}\right)^{7/3}$$

where subscript $_0$ refers to initial values.

Is the total energy $E + \Omega$ decreasing or increasing? Where is the energy going to, or coming from?

8.22 A neutron star radiates by direct accretion onto its surface (i.e., no accretion disk is formed) at a rate set by the Eddington limit. Show that the temperature T of the resulting blackbody radiation from the neutron star surfaces varies as its mass M increases, according to

$$\frac{T}{T_0} = \left(\frac{M}{M_0}\right)^{5/12}.$$

8.23 (a) White dwarf material is almost perfectly conducting so that such stars can be taken as having a uniform temperature T. If a white dwarf has mass M and radius R and is taken to be made up of protons of mass m_p, and electrons of very small mass, each with thermal energy $\frac{3}{2}kT$, use Stefan's law to show that radiation will cool a white dwarf on a timescale of about

$$\tau_{cool} \simeq \frac{3Mk}{4\pi m_p \sigma T^3 R^2},$$

where k and s are the Boltzmann and Stefan-Boltzmann constants. Hence

(i) Calculate the cooling time for a white dwarf of mass 4×10^{30} kg and radius 2×10^6 m, at temperature $T = 10^7$ K.

(ii) Show that two white dwarfs of masses M_1, M_2 and of temperatures T_1, T_2 have cooling times in the ratio

$$\frac{\tau_{cool1}}{\tau_{cool2}} = \left(\frac{M_1}{M_2}\right)^{5/3} \left(\frac{T_2}{T_1}\right)^3,$$

and luminosity L_1, L_2 in the ratio

$$\frac{L_1}{L_2} = \left(\frac{M_2}{M_1}\right)^{2/3} \left(\frac{T_1}{T_2}\right)^4.$$

(b) A more exact description of the cooling described above is to consider the differential equation which says that rate of charge of thermal energy equals the rate of radiative output. That is

$$\frac{d}{dt}\left(\frac{3MkT}{m_p}\right) = \frac{3Mk}{m_p}\frac{dT}{dt} = -4\pi R^2 \sigma T^4,$$

where M, R are constant. Show that the resulting time evolution of the temperature $T(t)$ is then given by

$$\frac{T(t)}{T(0)} = \frac{1}{(1 + 3t/\tau_{cool})^{1/3}},$$

where τ_{cool} is as given in the first part of the question for $T = T_0$.

8.24$_e$ What are the main observational features of *pulsars*? [2]

Describe briefly the accepted model of pulsars and explain the origin of their high magnetic fields and spin rates. Why do they produce high energy particles ? [3]

8.25$_e$ What is meant by the term *black hole*? Name, and obtain an expression for, the effective radius of a black hole of mass M. [2]

Name, and give the value of, the mass above which cold stellar remnants are expected to form black holes. What evidence is there for the existence of black holes? [3]

8.26e Why are gravitationally accreting neutron stars proposed to explain some powerful compact X-ray sources? What provides the accreting mass and by what two mechanisms can this mass be transferred to the neutron star? [6]

Write down an expression for the accretion luminosity for a neutron star of mass M and radius R, and derive the Eddington radiation pressure limit to this luminosity. [4]

If the neutron star were to accrete the mass directly onto its surface (no accretion disk) show, using Stefan's law, that the steady surface temperature, T, would be

$$T = \left[\frac{GM\dot{M}}{4\pi R^3 \sigma} \right]^{1/4}$$

[4]

and evaluate T for $M = 4 \times 10^{30}$ kg, $R = 10^4$ m, $\dot{M} = 10^{-6} M_{\odot}\,\mathrm{yr}^{-1}$. [3]

[$1\mathrm{yr} = 3.2 \times 10^7$ sec]

9 Cosmology

9.1e State Hubble's law for the expansion of the Universe. A galaxy is observed to have a redshift of 0.03. Estimate its distance in Mpc given a Hubble constant of $65\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. [5]

9.2e Explain what is meant by the term *cosmological dark matter*. What evidence is there for the existence of such matter? [5]

9.3e Describe what is meant by a *standard candle* method for the determination of a distances in the Universe. Describe how the *period-luminosity* law as obtained from studies of Cepheid variable stars is used to determine distances. [5]

9.4e State three observational facts that support the hot big bang model. What is meant by the term recombination? [5]

9.5e State two main features for each of the following:

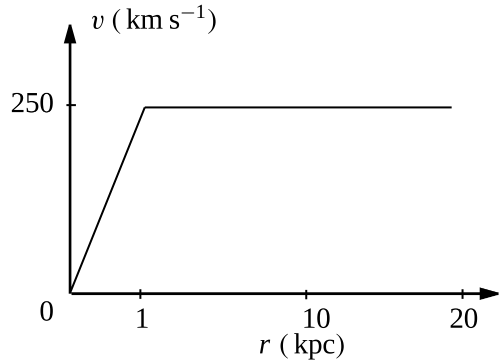
(a) spiral galaxies

(b) elliptical galaxies

(c) quasars. [5]

9.6 Elliptical galaxies are characterised as 'En', where $n = 10(1 - b/a)$ and a and b are the major and minor axes of the ellipse as seen on the sky. Calculate the ratio b/a for elliptical galaxies described as E0, E3 and E7, and sketch approximately to scale the appearance of such galaxies.

9.7 The rotation curve of a typical spiral galaxy can be approximated as shown in the diagram below.



Here v is the speed of rotation of matter about the centre of the galaxy (assuming circular motion), and r is the distance from the centre. From the graph, calculate the period of rotation of stars situated at 1 kpc, 10 kpc and 20 kpc from the centre of this galaxy. Express your answers in years. Suppose that the visible extent of the galaxy is 15 kpc in radius. What type of observations can be used to extend the rotation curve beyond the visible extent of the galaxy? Briefly comment on what conclusions can be drawn about the distribution and extent of matter in the galaxy from the shape of the rotation curve. (l year $= 3.16 \times 10^7$ s)

9.8 Explain what is meant by a rotation curve for a spiral galaxy and how it is measured. [4]

If the Sun is taken to be moving in a keplerian orbit of radius 8 kpc about the centre of our Galaxy with a velocity of 200 km s^{-1}, estimate the mass of the galaxy within the Sun's orbit

(a) in kilograms {1.3×10^{41} kg}

(b) in solar masses. {$0.7 \times 10^{11}\,M_\odot$}

State any assumptions you make. [8]

Sketch the rotation curve of a typical spiral galaxy and explain why this could provide evidence for the existence of dark matter. [8]

9.9 The redshifts of quasars and QSOs extend from $z \sim 0.05$ to $z \sim 4$. Using the redshift/velocity relationship, calculate the velocities of recession of quasars of redshift 0.05 and 4. Note that you will need to use the relativistic form of the relationship for the high value of z. (See notes)

Using Hubble's Law, calculate a value for the distance to these quasars. Express your answer in Mpc.

We measure on Earth the flux, F, from an astronomical object, where flux is in units of W m^{-2} or equivalent. To find intrinsic luminosity, L, of the object (in watts or equivalent) we assume that the luminosity spreads out equally in all directions. Show that this implies that $L = 4\pi r^2 F$, where r is the distance to the object. Hence calculate the optical luminosity of a quasar situated at a redshift of $z = 0.05$ whose optical flux is 10^{-11} W m^{-2}. Express your answer in units of $L_\odot$. (Hubble's constant, $H = 65$ km s^{-1} Mpc^{-1})

9.10 Broad and strong emission lines at 663 nm, 491 nm and 438 nm are observed in a starlike object. Explain why these could be interpreted as due to Hα, Hβ, and Hγ emission in a quasar. [4]

Estimate the redshift of the quasar. {0.01} [4]

Assuming Hubble's constant to be $65\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ calculate the distance to the quasar in Mpc. {46.2 Mpc} [4]

Calculate the optical luminosity of the quasar if its optical flux is $10^{-11}\,\mathrm{W\,m^{-2}}$. {$2.2 \times 10^{38}$ W} [4]

Assuming that this luminosity is produced by the accretion of matter into a black hole, find a lower limit on the mass accreted per year. {$0.02\,M_\odot$} [4]

(You may assume Hα is observed in laboratory at 656.3 nm)

9.11 Describe two methods for obtaining a distance estimate for a galaxy whose approximate distance is 20 Mpc. [4]

The period mean luminosity relation for cepheid variable population I stars is given approximately by

$$M = -2.6 - 3.8 \log_{10} P,$$

where M is the absolute magnitude and P the period in days. A cepheid star is observed in a neighbouring galaxy to have a period of 20 days, and an apparent magnitude of 19. What is the distance to the neighbouring galaxy? {2 Mpc} [10]

Assuming that the error bar on the constant term is ± 0.2, but the value of 3.8 is exact, what are the error bars on the distance? {± 0.2 Mpc} [6]

9.12 Assuming that a supercluster of galaxies contains around 10^{15} solar masses, and the average separation of superclusters is 100 Mpc, estimate the mean mass density of the Universe, and compare this with $\rho_c = 3H_0^2/(8\pi G)$.

Describe some of the difficulties involved in determining the mean mass density of the Universe. (Take $H = 65\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$)

9.13 Verify that the energy equation of Newtonian cosmology, for the case of critical density, can be written in the form

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \left(\frac{2GM}{R}\right)^{1/2}.$$

Here M is the constant mass inside a sphere of galaxies of radius R, which expands with the Universe.

Show by integration, using the big bang condition $R = 0$ at $t = 0$, that the present age of the Universe, for the case of critical density, is

$$t = \frac{2}{3H},$$

where H is the present value of Hubble's constant defined by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = HR.$$

9.14e Describe the origin of the cosmic background radiation (CMB) according to the hot big bang model. [4]

Explain the significance of the anisotropies observed in the cosmic background radiation for the subsequent formation of large scale structures in the Universe. [3]

The scale factor, $R(t)$, in a big bang cosmological model is given as a function of cosmic time, t, by $R(t) = at^{2/3}$, where a is a constant. Explain why in this model the matter density of the early Universe would have been extremely high. [2]

Assuming that the energy density, u, of the CMB radiation varies as $R(t)^{-4}$, calculate the redshift, z, at which recombination took place, taking the present day temperature of the CMB to be 3 K. Taking the present age of the Universe to be 1.5×10^{10} years, calculate the time at which recombination took place. (You may assume that recombination took place at a temperature of 3 000 K.) [8]

9.15e State Hubble's law for the expansion of the Universe. The value of the Hubble constant is still uncertain. What reasons are there for this?

The distance, $d(t)$, between any comoving galaxies at any cosmic time, t, is given by $d(t) = R(t)/R_0 \times d_0$, where d_0 is the distance at the present epoch. If $R(t) = R_0\,(t/t_0)^{2/3}$, show that Hubble's constant is given by $H_0 = \frac{2}{3t_0}$.

If the mean distance between superclusters of galaxies today is 20 Mpc, what was the mean distance at redshift time $t = t_0/8$? What is the redshift corresponding to this time?

9.16e What are meant by the terms

(a) galaxy cluster

(b) supercluster? [4]

Explain what is meant by the critical matter density. [5]

An average supercluster contains about 10^{15} solar masses. Taking the mean separation between superclusters to be about 100 Mpc, estimate the mass density of the Universe in kg m^{-3}. Hence *estimate* the value for the ratio of matter density to the critical mass density, Ω. [6]

State two observational facts that indicate the presence of significant amounts of dark matter in the Universe. [2]

[You may assume that $\rho_{\text{crit}} = 3H_0^2/(8\pi G)$, $H_0 = 65\,\text{kms}^{-1}\text{Mpc}^{-1}$, $G = 6.7\times10^{-11}\,\text{N m}^2\,\text{kg}^{-2}$, and one solar mass is 2×10^{30} kg.]

9.17 In a big bang model the scale factor $R(t)$ is given by $R(t) = R_0 t^{2/3}$.

(a) If the average distance between galaxy clusters is now 20 Mpc , what would it have been when the Universe was half its present age?

(b) show that Hubble's constant in this model is given by $H = \frac{2}{3t}$. What is the age of the Universe with a Hubble constant of 65 km s^{-1} Mpc^{-1}?

(c) Evaluate the critical density at the present epoch, given that

$$\rho_c = \frac{3H_0^2}{8\pi G},$$

where $G = 6.7 \times 10^{-11}\,\text{N kg}^{-2}\,\text{m}^{-2}$ is the universal gravitational constant.

The matter density in the Universe varies as

$$\frac{\rho}{\rho_0} = \left(\frac{R_0}{R}\right)^3,$$

and the energy density of radiation as

$$\frac{u}{u_0} = \left(\frac{R_0}{R}\right)^4,$$

Sketch ρ/ρ_0, u/u_0 and u/ρ as a function of time.

(d) Will there be a time in the Universe's past when the radiation energy density was greater than the matter density? If so, how would you go about calculating it?

Assume that the present matter density (including dark matter) is 0.2 of the critical density. Hence the present ratio of radiation to matter density is about 10^{-4}.

[Hint: The energy density of blackbody radiation is given by $u = aT^4$ where $a = 7.57 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$ hence the energy density of the radiation is given by $u = 7.57 \times 10^{-16}(3\,\mathrm{K})^4\,\mathrm{J\,m^{-3}\,K^{-4}}$ or about $6 \times 10^{-15}\,\mathrm{J\,m^{-3}}$ (You need to convert this to $\mathrm{kg\,m^{-3}}$)].

(e) what was the age of the Universe at recombination, if this took place at $T = 4\,000\,\mathrm{K}$? Was this before radiation became dominant or after in the life of the Universe?

Questions:
How does Ω vary with cosmic time this model?
Do you believe in the big bang?

For the very keen - If the scale factor was given by $R(t) = R_0 t^{\alpha}$, where $\alpha > 0$, how would the above calculation differ?

9.18e State how an Sa and Sc spiral galaxy differ in appearance. If the apparent major and minor axes of an elliptical galaxy subtend angles of 0.5 arcmin and 0.2 arcmin respectively, what is the Hubble classification of this galaxy? [5]

9.19e Explain what is meant by the *cosmological principle*. Give two pieces of observational evidence which support the validity of the cosmological principle. [5]

9.20e Explain how HST observations of Cepheid variables in other galaxies have significantly improved recent estimates of the Hubble constant, H_0. [5]

Give two reasons why type Ia supernovae (SN Ia) are useful standard candles for measuring extragalactic distances. [2]

A SNIa, SN1998a, is observed to have a B band apparent magnitude at maximum light of $m_{\mathrm{B}}(\mathrm{max}) = 10.3$. Cepheid observations in the SNIa host galaxy give an estimated distance modulus for the galaxy of $\mu = 30.0$. What is the estimated distance of the galaxy, in Mpc? What is the estimated B band absolute magnitude of the SNIa at maximum light? [5]

From a sample of distant SNIa the following relation is found between $m_{\mathrm{B}}(\mathrm{max})$ and the recession velocity, v, in $\mathrm{kms^{-1}}$, of the host galaxies:

$$m_{\mathrm{B}}(\mathrm{max}) = 5\log_{10} v - 3.6$$

By substituting Hubble's Law into the above relation, obtain an estimate of H_0 from the apparent magnitude and estimated distance of SN1998a. [5]

9.21e Explain briefly why quasars are believed to be the cores of very young galaxies. What is generally thought to be their source of energy? [5]

A quasar is observed to have a redshift of $z = 3.5$. Using the relativistic Doppler formula,

$$z = \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1$$

calculate the recession velocity, v, in $\mathrm{km\,s^{-1}}$, of the quasar. [6]

If the Hubble constant, $H_0 = 65\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, calculate the distance to the quasar in Mpc. [2]

If the brightness of the quasar changes over a timescale of one week, what is the maximum possible diameter, in AU, of the light emitting region? [4]

9.22 A cluster containing 1 000 galaxies has a characteristic radius of 10 Mpc and an r.m.s. radial peculiar velocity of $1\,000\,\mathrm{km\,s^{-1}}$ at this radius from the centre of the cluster. Using the formula given in the lectures, calculate a virial estimate of the mass of the cluster, expressing your answer in solar masses.

If the average number of stars per galaxy is 10^{11}, and the average stellar mass is approximately one solar mass, calculate the mass of the cluster in the form of stars. Comment on your answer.

9.23 State two properties which a good *standard candle* distance indicator should possess. Give an example of a standard candle distance indicator used to measure galaxy distances beyond the Local Group. [4]

The period mean luminosity relations for Cepheid variables in the V and I wavelength bands are given approximately by

$$M_V = -2.76 \log_{10} P - 1.40$$

$$\text{and} \quad M_I = -3.06 \log_{10} P - 1.81$$

where P is the period of the Cepheid in days, and M_V and M_I are its V band and I band mean absolute magnitudes respectively. A Cepheid is observed in a distant galaxy to have a period of 35 days, and a mean V band and I band apparent magnitudes of $m_V = 24.9$ and $m_I = 23.6$ respectively. Estimate the distance of the galaxy using first the V band and then the I band Cepheid relation. [12]

Suggest an explanation why the V band and I band distance estimates are different. [4]

9.24 Explain what is meant by saying that the Universe is *homogeneous* and *isotropic*. Above what approximate scale do galaxy redshift surveys begin to display homogeneity and isotropy? [4]

Write down the relation between the *proper distance*, $r(t)$, and *comoving separation*, s, of two galaxies at time, t, in terms of the scale factor, $R(t)$. Show how this relation implies a linear Hubble law between recession velocity and proper distance. [4]

In a critical density Universe the time evolution of the scale factor is described by $R(t) = R_0(t/t_0)^{2/3}$ (where the subscript '0' denotes the present day value). Show that this equation implies that the Hubble constant, H_0, and the present age of a critical density Universe, t_0, are related by $H_0 = \frac{2}{3} t_0^{-1}$. [5]

Show that the following relation holds between redshift and time for a critical density Universe $z = (t/t_0)^{-2/3} - 1$. [3]

At what redshift would a galaxy be observed if its light was emitted when the Universe was one tenth of its current age? What was the value of the scale factor at this epoch, in terms of its current value? [4]

9.25e Explain what is meant by the term *cosmological distance ladder*.

Describe one method of determining the distances of galaxies out to 3 Mpc, and another for distances beyond 50 Mpc. Explain why such measurements cannot be considered to be very accurate. [10]

D.A.D.
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